Optimization of elastic circular plates with additional supports

JAAN LELLEP
University of Tartu
Institute of Mathematics
J. Liivi Street 2, 50409 Tartu
ESTONIA
jaan.lellep@ut.ee

JULIA POLIKARPUS
University of Tartu
Institute of Mathematics
J. Liivi Street 2, 50409 Tartu
ESTONIA
julia.polikarpus@ut.ee

Abstract: Elastic circular plates with additional rigid ring supports are investigated. It is assumed that the plate is made of an ideal elastic material obeying the Hooke's law. The plate is simply supported at the edge and it is resting on an absolutely rigid ring support. The problem of optimal location of the internal support is solved under the condition that the cost of the internal support is proportional to its length. The problem is treated as a particular problem of optimal control. The variational methods of the optimal control theory are used in order to get necessary conditions of optimality. Numerical results are presented for the case of uniformly distributed tranverse pressure.

Key–Words: circular plate, additional support, optimization, optimal control, thin walled plates

1 Introduction

One of the ways of increasing the stiffness of beams, plates and shells is to furnish these structural elements with additional supports. Evidently, it is reasonable to settle these supports in the optimal positions.

The problem of minimization of the compliance of elastic beams and the determination of the optimal location of the additional support was first formulated by Mroz and Rozvany [15]. In the paper [15] designs of minimum compliance of beams are established in the case of quasistatic loading. Later Szelag and Mroz [20], Akesson and Olhoff [1] treated the problems of maximal eigenfrequency for given stiffness with respect to the location of the additional support. Bojczuk and Mroz [3] developed a new method for simultaneous optimization of topology, configuration and cross-sectional dimensions of elastic beams and beam structures extending earlier results by Garstecki and Mroz [6], Mroz and Lekszycki [14], also by Lepik [13]. In the subsequent papers by Bojczuk and Mroz [4] this concept was applied for optimal design of active supports with force actuators. Olhoff and Akesson [16] treated the stability of columns.

A lot of attention has been paid in the literature to the optimization of internal supports to beam, plate and shell structures in the case of inelastic materials. Probably the first paper in this area is due to Prager and Rozvany [17]. Systematic reviews of results obtained in earlier papers are presented by Rozvany [19], also by Lellep and Lepik [10]. Optimal designs of circular cylindrical shells with additional supports are established by Lellep [8, 9] in the case of an ideal plastic material. The behaviour of geometrically non-linear cylindrical shells with internal supports is studied in [11].

A design sensitivity analysis for the deflection of beam or plate structures was undertaken by Wang [23] in the case of simple supports located at given mesh nodes.

In the present paper an analytical method of determination of positions of rigid ring supports for circular plates is developed. The analysis is confied to the axisymmetric response of elastic plates to subjected loads.

2 Formulation of the problem

As we are studying the axisymmetric response of the plate all points lying at the circle with radius r have common displacements $W(r)$ in the transverse direction as well as common deformations and curvatures κ_1, κ_2 in the radial and circumferential directions, respectively. Note that the radial displacement, also radial and circumferential membrane forces will be neglected in the present study whereas classical equations of the bending theory of thin plates will be used.

The plate under consideration is simply supported at the edge and it is resting on an absolutely rigid ring support of unknown radius $r = s$. From practical considerations it is evident that the desirable position of the additional support is such that the maximal deflection of the plate is as small as possible. Thus the optimal location of the internal support should minimize

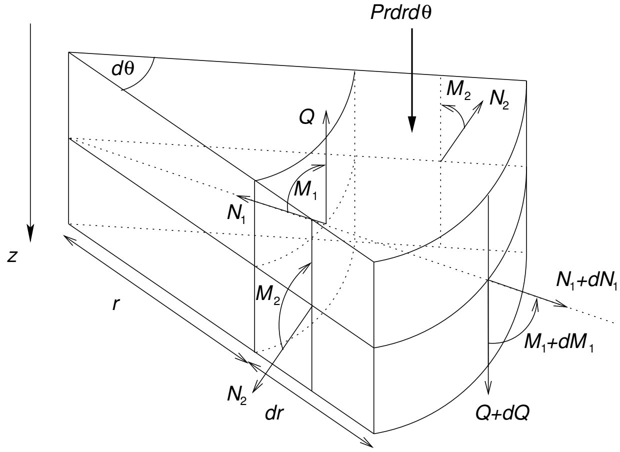

Figure 1: An element of the circular plate.

the functional

$$J_1 = \max_{r\in[0,R)} W(r,P,s) \tag{1}$$

for given loading $P = P(r)$ and thickness $h = h(r)$. However, the cost function presented in the form (1) has several drawbacks. First of all, it is a non-differentiable and non-additive functional. The use of non-differentiable functionals in the solution of problems of optimization is quite complicated. On the other hand, the functional (1) ignores the expenditures necessary for manufacturing of the additional support.

It can be shown that an approximation of the functional (1) can be presented as [2, 10]

$$J_2 = \left(\int_0^R W^k r dr\right)^{\frac{1}{k}} \tag{2}$$

where k is an integer. If $k \to \infty$ then $J_2 \to \|W\|$.

Due to the circumstancies mentioned above in the present paper the cost function

$$J = \int_0^R W^k r dr + \mu_0 2\pi s \tag{3}$$

will be employed. In (3) μ_0 stands for the specific cost (cost per unique length) of the additional support. We assume herein that the material cost of the additional support is proportional to its length.

The aim of the paper is to determine the design of the plate with an additional support which minimizes the cost function (3) so that at each value of P governing equations of the theory of thin axisymmetric plates with appropriate boundary conditions are satisfied.

3 Governing equations

In the present paper we shall employ the linear theory of thin plates (see Reddy [18], Vinson [22]). According to this approach one can treat the equilibrium of internal and external forces and couples on the basis of an undeformed element of the plate. Let M_1, M_2 be the generalized couples called bending moments in the radial and circumferential directions, respectively. Bending moments M_1, M_2 are the only generalized stress components contributing to the internal energy. Note that the membrane forces are assumed to be small so that one can neglect the membrane action of internal forces. Although the shear force Q may be finite it does not contribute to the internal energy in the classical plate theory. The reason is that the corresponding strain component vanishes.

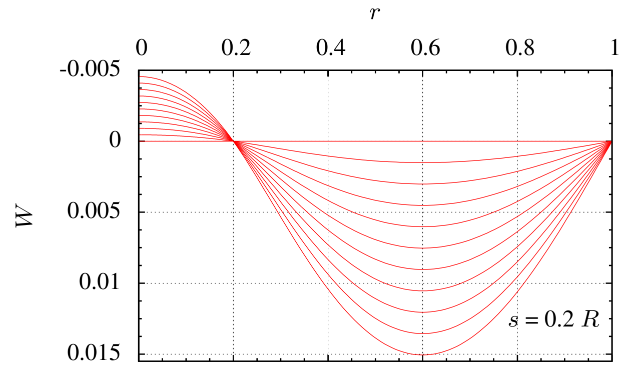

Figure 2: Transverse deflections.

In the frameworks of the classical plate theory couples M_1, M_2 with forces Q and P form a system of forces and moments which keep the element of the plate in equilibrium. The equilibrium conditions of the plate element presented in Fig. 1 can be written as

$$\frac{d}{dr}(rM_1)-M_2-rQ = 0, \frac{d}{dr}(rQ)+P(r)r = 0. \tag{4}$$

The system of governing equations can be presented as [12]

$$\begin{aligned}
\frac{dW}{dr} &= Z, \\
\frac{dZ}{dr} &= -\frac{M_1}{D} - \frac{\nu Z}{r}, \\
\frac{dM_1}{dr} &= \frac{D(\nu^2-1)Z}{r^2} - \frac{M_1(1-\nu)}{r} + Q, \\
\frac{dQ}{dr} &= -\frac{Q}{r} - P(r),
\end{aligned} \tag{5}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{6}$$

is the flexural stiffness. Variables W, Z, M_1, Q will be treated as state variables which satisfy the state equations (5) with appropriate boundary and intermediate conditions. At the outer edge of the plate , e. g. at $r = R$ bending moment M_1 and the deflection W must vanish. Thus

$$M_1(R) = 0, \qquad W(R) = 0. \tag{7}$$

Due to the symmetry at the center of the plate

$$\frac{dW}{dr}(0) = 0, \qquad Q(0) = 0. \tag{8}$$

At $r = s$ where the rigid ring support is located must be

$$W(s) = 0. \tag{9}$$

Note that state variables W, Z, M_1 are continuous whereas Q can be discontinuous at $r = s$.

4 Necessary optimality conditions

In order to establish the requirements to be satisfied by the optimal solution let us introduce the augmented functional (see Bryson [5], Hall [7]; Lellep, Polikarpus [12])

$$J_* = \mu s + \int_0^s F_* dr + \int_s^R F_* dr \tag{10}$$

where according to (3), (5)

$$\begin{aligned} F_* = W^k + \psi_1\left(\frac{dW}{dr} - Z\right) + \\ +\psi_2\left(\frac{dZ}{dr} + \frac{M_1}{D} + \frac{\nu Z}{r}\right) + \\ +\psi_3\left(\frac{dM_1}{dr} - \frac{D(\nu^2-1)Z}{r^2} + \right. \\ \left. + \frac{M_1(1-\nu)}{r} - Q\right) + \\ +\psi_4\left(\frac{dQ}{dr} + \frac{Q}{r} + P(r)\right) \end{aligned} \tag{11}$$

and $\mu = 2\pi\mu_0$, the quantities $\psi_1 - \psi_4$ being adjoint variables.

Evidently the problem posed above belongs to the class of optimal control problems with moving boundaries. Therefore, one has to employ total variations when deriving necessary conditions of minimum of the functional (10). The total variation of a state variable y at $r = s + 0$ or at $r = s - 0$ must be calculated by the following sample

$$\Delta y(s \pm 0) = \delta y(s \pm 0) + \frac{dy(s \pm 0)}{dr} \cdot \Delta s \tag{12}$$

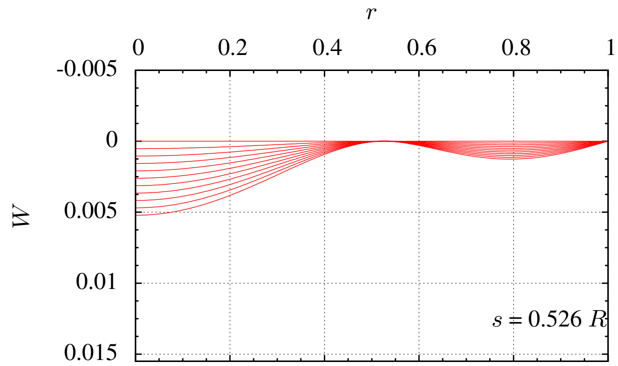

Figure 3: Deflections for the optimal case.

where Δy is the total variation and δy stands for the ordinary variation of the variable y. If the state variable is continuous at $r = s$ then, ofcourse, $\Delta y(s - 0) = \Delta y(s+0) = \Delta y(s)$. However, in the case of discontinuous variables one has to distinguish the quantities $\Delta y(s - 0)$ and $\Delta y(s + 0)$. Note that even in the case of continuous variables the quantities $\delta y(s - 0)$ and $\delta y(s + 0)$ must not be equal to each other.

Making use of (10) – (12) one easily obtains from the equation $\Delta J_* = 0$ the system of adjoint equations

$$\begin{aligned} \frac{d\psi_1}{dr} &= rkW^{k-1}, \\ \frac{d\psi_2}{dr} &= -\psi_1 + \frac{\nu\psi_2}{r} - \frac{D(\nu^2-1)\psi_3}{r^2}, \\ \frac{d\psi_3}{dr} &= \frac{\psi_2}{D} + \frac{\psi_3(1-\nu)}{r}, \\ \frac{d\psi_4}{dr} &= -\psi_3 + \frac{\psi_4}{r}. \end{aligned} \tag{13}$$

Note that although the adjoint set (13) holds good for each $r \in [0, r]$ it must be integrated separately in regions $(0, s)$ and (s, R), respectively. The reason is that some of adjoint variables can be discontinuous at $r = s$.

Boundary conditions (7), (8) admit to present the transversality conditions as

$$\psi_1(0) = 0, \qquad \psi_3(0) = 0 \tag{14}$$

and

$$\psi_2(R) = 0, \qquad \psi_4(R) = 0. \tag{15}$$

Equations (13) with (14), (15) admit to rewrite the equation $\Delta J_* = 0$ as

$$\begin{aligned} \mu\Delta s - (\psi_1\delta W + \psi_2\delta Z + \\ +\psi_3\delta M_1 + \psi_4\delta Q)|_{s-0}^{s+0} = 0. \end{aligned} \tag{16}$$

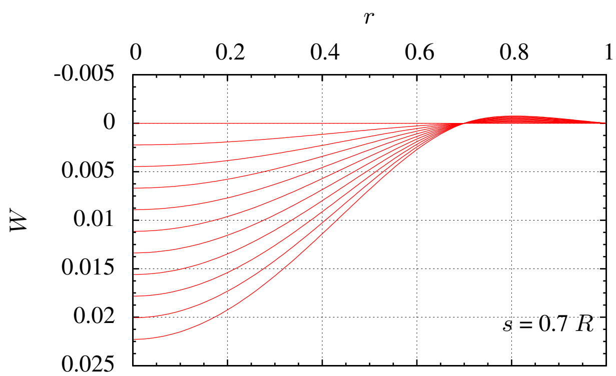

Figure 4: Transverse deflections.

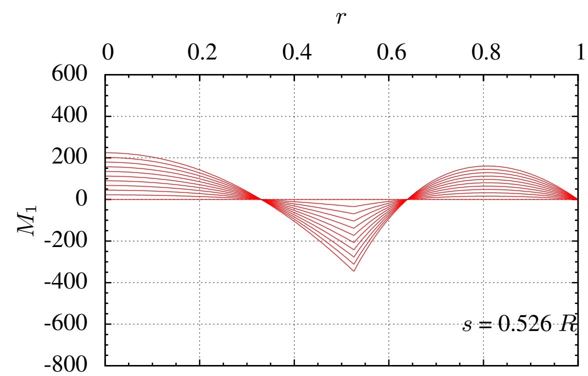

Figure 5: Bending moment.

From the physical considerations it is evident that W, Z and M_1 are continuous at $r = s$. Thus following the scheme (12) one can write

$$\begin{aligned} &\psi_2(s-0) - \psi_2(s+0) = 0, \\ &\psi_3(s-0) - \psi_3(s+0) = 0 \end{aligned} \tag{17}$$

and

$$\psi_4(s-0) = \psi_4(s+0) = 0. \tag{18}$$

It was assumed above that Z and M_1 are continuous everywhere; thus in particular at $r = s$. Bearing in mind the continuity of M_1 it is clear that $\kappa_1 = -\dfrac{dZ}{dr}$ is also continuous at $r = s$.

Substituting (17) – (18) in (16) and taking into account the continuity of Z, κ_1, κ_2 and ψ_2, ψ_3, also the arbitrariness of the increment Δs one can present (16) as

$$\mu + [\psi_1(s)]\frac{dW(s)}{dr} + \psi_3(s)\left[\frac{dM_1(s)}{dr}\right] = 0. \tag{19}$$

In (19) the quadratic brackets denote the finite jumps of corresponding variables at $r = s$, e. g.

$$[y(s)] = y(s+0) - y(s-0)$$

where $y(s \pm 0)$ stands for right and left hand limits of the discontinuous variable $y(r)$ at $r = s$.

5 Solution of governing equations

Consider the solution of state equations (5) in greater detail in the case when the plate thickness h is constant. In this case it follows from (6) that $D = const$, as well. Integrating the last equation in the system (5) one obtains

$$Q = -\frac{1}{r}\left(\int P(r)dr + C_{\pm}\right) \tag{20}$$

where C_+ and C_- stand for constants of integration in the regions $[0, s]$ and $[s, R]$, respectively.

For the subsequent integration of (5) it is reasonable to substitute Q and M_1 making use of (20). This results in a fourth order equation with respect to the deflection W known from the theory of elatic plates (see Reddy [18], Vinson [22]; Ventsel, Krauthammer [21]). The general solution of this equation can be presented in the case $P = const$ as

$$\begin{aligned} W = &\frac{Pr^4}{64D} + A_{1j}r^2\ln r + A_{2j}r^2 + \\ &+ A_{3j}\ln r + A_{4j} \end{aligned} \tag{21}$$

for $r \in [r_j, r_{j+1}]$ and $j = 0, 1$. Here the following notation is used: $r_0 = 0$, $r_1 = s$ and $r_2 = R$. Evidently,

$$Z = \frac{Pr^3}{16D} + A_{1j}r(2\ln r + 1) + 2A_{2j}r + \frac{A_{3j}}{r} \tag{22}$$

and

$$\begin{aligned} M_1 = &-\frac{Pr^2(3+\nu)}{16} - \\ &- A_{1j}D[3 + \nu + 2(1+\nu)\ln r] - \\ &- 2DA_{2j}(1+\nu) - \frac{D(\nu-1)}{r^2}A_{3j}, \\ M_2 = &-\frac{Pr^2(1+3\nu)}{16} - \\ &- A_{1j}D[1 + 3\nu + 2(1+\nu)\ln r] - \\ &- 2DA_{2j}(1+\nu) - \frac{D(\nu-1)}{r^2}A_{3j}. \end{aligned} \tag{23}$$

The integration constants A_{1j} – A_{4j} will be determined from the boundary and continuity conditions.

6 Discussion of results

Results of calculations are presented in Fig. 2 – 5. The calculations are implemented for $k = 1$ and $\mu = 0$.

In Fig. 2 – 4 the distributions of deflections of the plate are presented for various values of the transverse load intensity. Fig. 2 and Fig. 4 correspond to the positions of the support at $s = 0.2R$ and $s = 0.7R$ whereas Fig. 3 is associated with the optimal location of the intermediate support. The optimal solution corresponds to $s = 0.526R$. It can be seen from Fig. 2 that in the case of smaller values of the radius of the intermediate support deflections at the central part of the plate for $r < 0.2R$ are directed upward despite the pressure is directed downward. Similarily in the case when $s = 0.7R$ one can see negative deflections in the outward region for $r > 0.7R$ (Fig. 4). However, in the case of optimal position of the additional support the deflections are non-negative everywhere (Fig. 3). It is somewhat surprising that the maximal deflections in the central and outward regions of the plate, respectively, are quite different in the optimal case. However, one has to take into account that the cost function (3) with $\mu_0 = 0$, $k = 1$ corresponds to the volume of the axisymmetric body.

In Fig. 5 bending moment M_1 is presented for the optimal case. It can be seen from Fig. 5 that the slope of the radial bending moment has finite jumps at the support position, as might be expected. It is some what surprising that the radial bending moment vanishes at an internal point for any values of the transverse pressure loading. It reveals from Fig. 5 that in the case of smaller values of the radius of the internal support the radial bending moment remains negative in the central part of the plate. It is negative in the vicinity of the support in the optimal case, as well.

7 Concluding remarks

Making use of the variational methods of the theory of optimal control the problem of optimal location of an additional rigid ring support for a circular plate was solved. The plate is made of an elastic material and subjected to a distributed transverse pressure. Necessary optimality conditions have derived under the assumption that the cost of the additional support is taken into account. Numerical results have presented for the plate simply supported at the edge and subjected to the uniformly distributed transverse pressure.

Calculations carried out showed that the optimal position of the additional support admits to diminish essentially the cost function. It revealed by calculations that the both, radial and circumferential bending moments are continuous over the entire plate.

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