Optimization of elastic circular plates with additional supports

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Outline

- Introduction
- Problem formulation
- Basic equations and concepts
- Constitutive equations
- Optimality conditions
- Numerical results
- Concluding remarks

Introduction

- Mroz, Rozvany (1975)
- Akesson, Olhoff (1988)

Annular plate

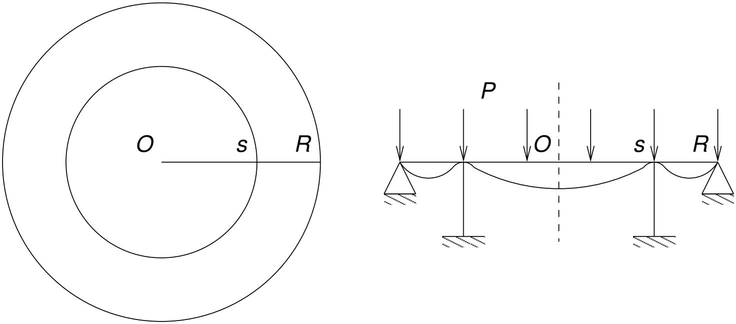


Figure: A circular plate with support.

Problem formulation

$$J = \int_0^R W^k r dr + \mu_0 2\pi s \tag{1}$$

The aim of the paper is to determine the design of the plate with an additional support which minimizes the cost function (1) so that at each value of P governing equations of the theory of thin axisymmetric plates with appropriate boundary conditions are satisfied.

Basic equations and concepts

- Equilibrium conditions

$$\frac{d}{dr}(rM_1) - M_2 - rQ = 0, \qquad \frac{d}{dr}(rQ) = -Pr$$

- Strain components

$$\kappa_1 = -\frac{d^2W}{dr^2}, \qquad \kappa_2 = -\frac{1}{r}\frac{dW}{dr}$$

- Hooke's law

$$M_1 = D(\kappa_1 + \nu\kappa_2), \qquad M_2 = D(\kappa_2 + \nu\kappa_1)$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

Constitutive equations

- Governing equations

$$
\begin{aligned}
\frac{dW}{dr} &= Z, \\
\frac{dZ}{dr} &= -\frac{M_1}{D} - \frac{\nu Z}{r}, \\
\frac{dM_1}{dr} &= \frac{D(\nu^2 - 1)Z}{r^2} - \frac{M_1(1-\nu)}{r} + Q, \\
\frac{dQ}{dr} &= -\frac{Q}{r} - P(r).
\end{aligned}
\tag{2}
$$

Constitutive equations

- Boundary conditions

$$M_1(R) = 0, \qquad W(R) = 0 \tag{3}$$

$$\frac{dW}{dr}(0) = 0, \qquad Q(0) = 0 \tag{4}$$

Optimality conditions

$$J_* = \mu s + \int_0^s F_* dr + \int_s^R F_* dr \tag{5}$$

$$\begin{aligned} F_* = {} & W^k + \psi_1\left(\frac{dW}{dr} - Z\right) + \psi_2\left(\frac{dZ}{dr} + \frac{M_1}{D} + \frac{\nu Z}{r}\right) + \\ & + \psi_3\left(\frac{dM_1}{dr} - \frac{D(\nu^2 - 1)Z}{r^2} + \frac{M_1(1-\nu)}{r} - Q\right) + \\ & + \psi_4\left(\frac{dQ}{dr} + \frac{Q}{r} + P(r)\right) \end{aligned} \tag{6}$$

Optimality conditions

$$\psi_2(s-0) - \psi_2(s+0) = 0,$$

$$\psi_3(s-0) - \psi_3(s+0) = 0, \tag{7}$$

$$\psi_4(s-0) = \psi_4(s+0) = 0,$$

$$\mu + [\psi_1(s)]\frac{dW(s)}{dr} + \psi_3(s)\left[\frac{dM_1(s)}{dr}\right] = 0. \tag{8}$$

Trasversality conditions

$$\psi_1(0) = 0, \qquad \psi_3(0) = 0 \tag{9}$$

$$\psi_2(R) = 0, \qquad \psi_4(R) = 0 \tag{10}$$

System of adjoint equations

$$
\begin{aligned}
\frac{d\psi_1}{dr} &= rkW^{k-1}, \\
\frac{d\psi_2}{dr} &= -\psi_1 + \frac{\nu\psi_2}{r} - \frac{D(\nu^2 - 1)\psi_3}{r^2}, \\
\frac{d\psi_3}{dr} &= \frac{\psi_2}{D} + \frac{\psi_3(1-\nu)}{r}, \\
\frac{d\psi_4}{dr} &= -\psi_3 + \frac{\psi_4}{r}.
\end{aligned}
\tag{11}
$$

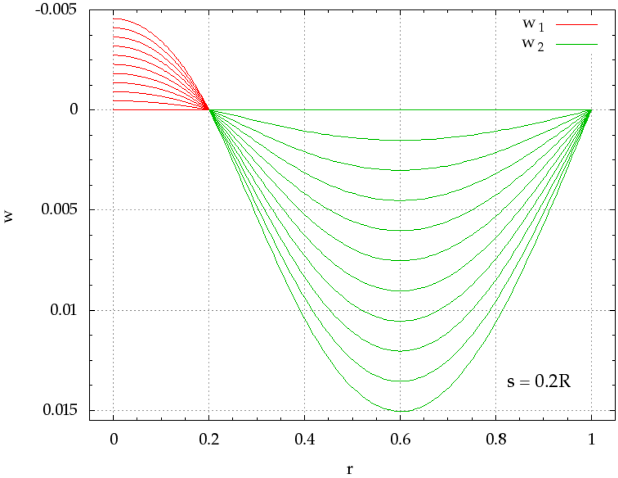
Solution of governing equations

$$
\begin{aligned}
W &= \frac{Pr^4}{64D} + A_{1j} r^2 \ln r + A_{2j} r^2 + A_{3j} \ln r + A_{4j},\\
Z &= \frac{Pr^3}{16D} + A_{1j} r(2\ln r + 1) + 2A_{2j} r + \frac{A_{3j}}{r},\\
M_1 &= -\frac{Pr^2(3+\nu)}{16} - A_{1j} D[3+\nu+2(1+\nu)\ln r] -\\
&-2DA_{2j}(1+\nu) - \frac{D(\nu-1)}{r^2} A_{3j},\\
M_2 &= -\frac{Pr^2(1+3\nu)}{16} - A_{1j} D[1+3\nu+2(1+\nu)\ln r] -\\
&-2DA_{2j}(1+\nu) - \frac{D(\nu-1)}{r^2} A_{3j}.
\end{aligned}
\tag{12}
$$

Solution of the adjoint system

$$
\begin{aligned}
\psi_1 &= \frac{r^2}{2} + C_{1j}, \\
\psi_2 &= C_{2j} + \frac{C_{3j}}{r} - \frac{(3+\nu)r^3}{16} - \frac{C_{1j}(1+\nu)r\ln r}{2}, \\
\psi_3 &= \frac{C_{2j}r^2}{D(\nu+1)} + \frac{C_{3j}}{D(\nu-1)} - \frac{r^4}{16D} - \\
&- \frac{C_{1j}r^2}{D(\nu^2-1)} + \frac{C_{1j}r^2[(1-\nu)\ln r + 1]}{2D(\nu-1)}, \\
\psi_4 &= -\frac{C_{2j}r^3}{2D(\nu+1)} - \frac{C_{3j}r\ln r}{D(\nu-1)} + C_{4j}r + \\
&+ \frac{r^5}{64D} + \frac{C_{1j}r^3\ln r}{4D} + \frac{C_{1j}(3-2\nu-\nu^2)r^3}{8D(\nu^2-1)}
\end{aligned}
\tag{13}
$$

Deflection



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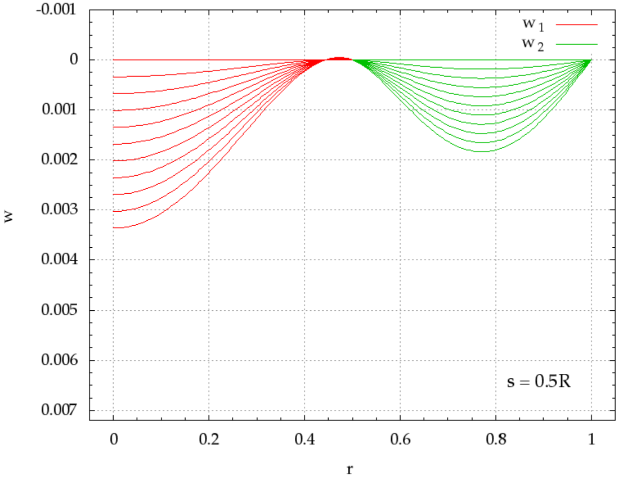
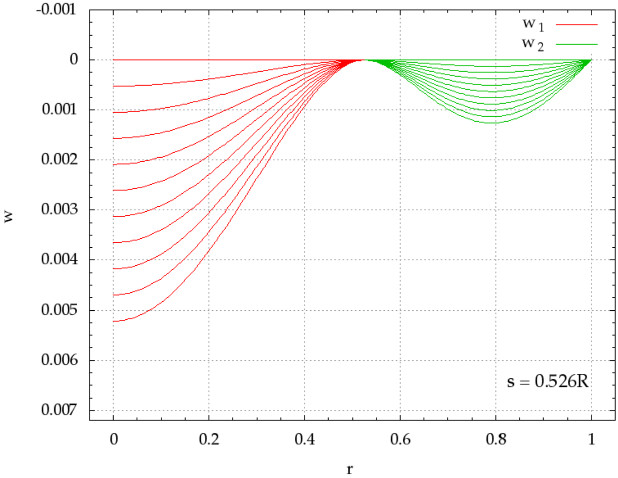


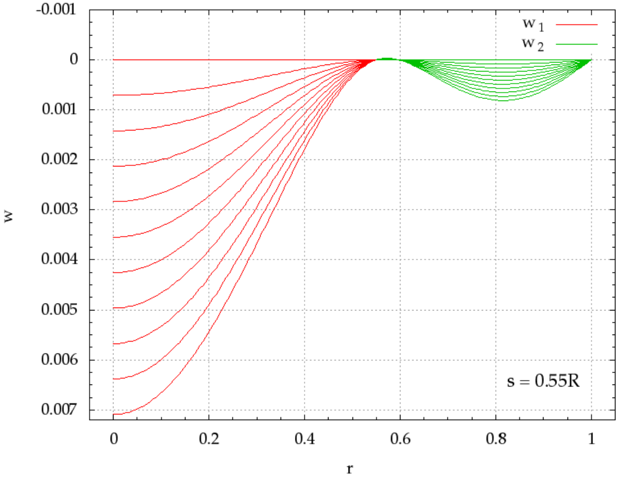
-0.001
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
w
0
0.2
0.4
0.6
0.8
1
r
w_1
w_2
s = 0.5R

Deflection



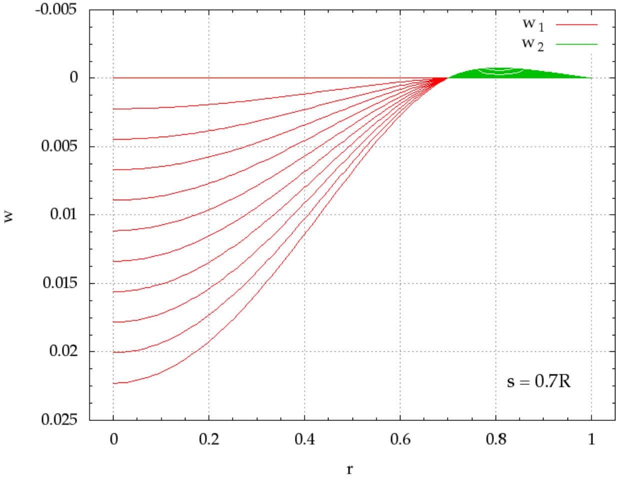
-0.001
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
w
0
0.2
0.4
0.6
0.8
1
r
w 1
w 2
s = 0.526R

Deflection

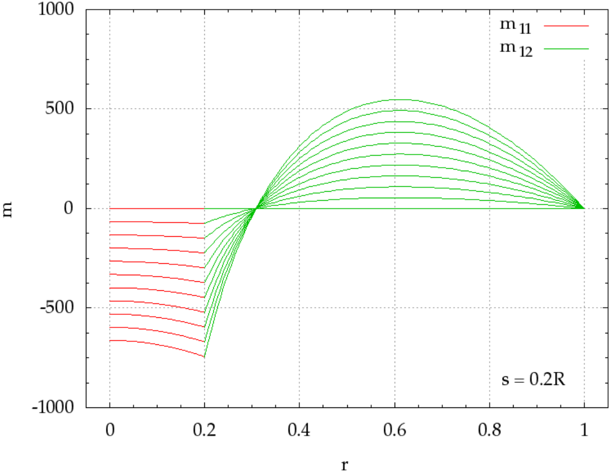


-0.001
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
w
0
0.2
0.4
0.6
0.8
1
r
w 1
w 2
s = 0.55R

Deflection



Moment



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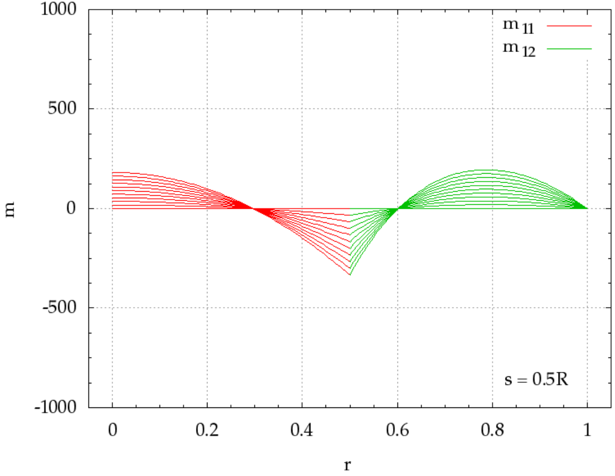

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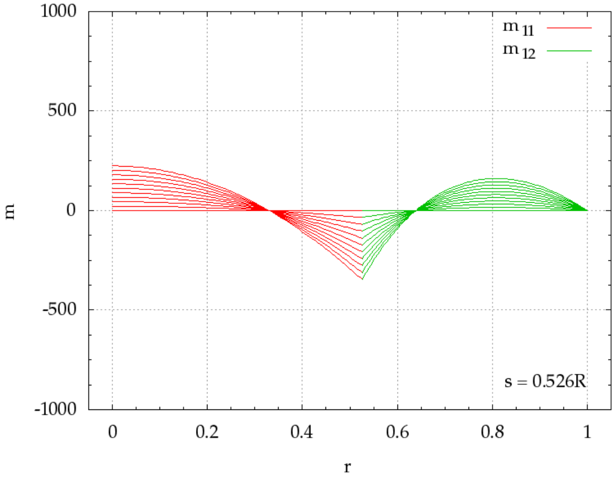

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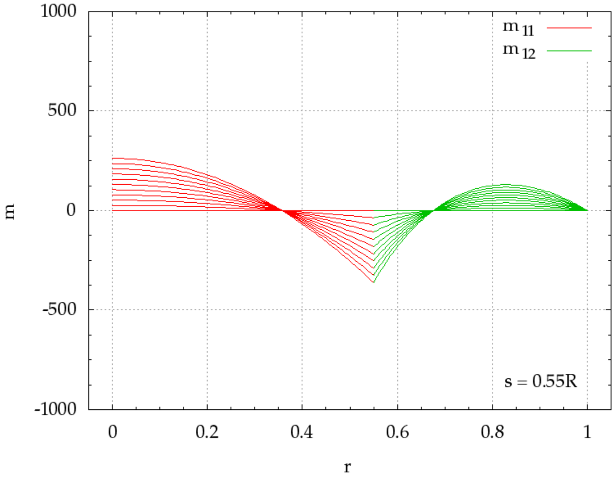

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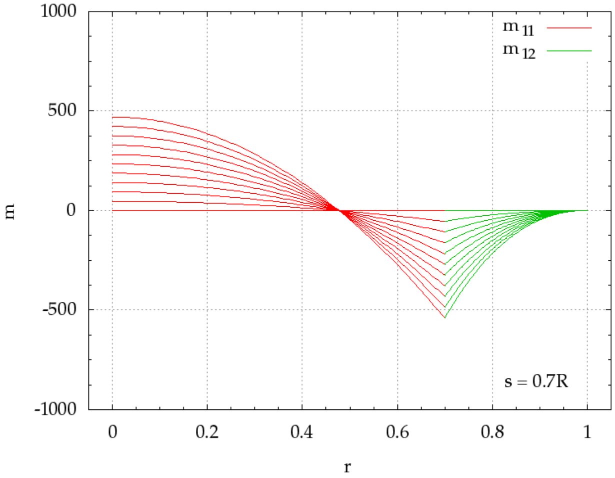

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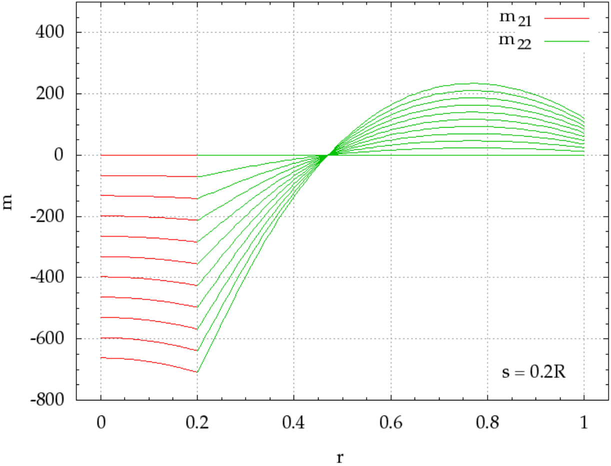

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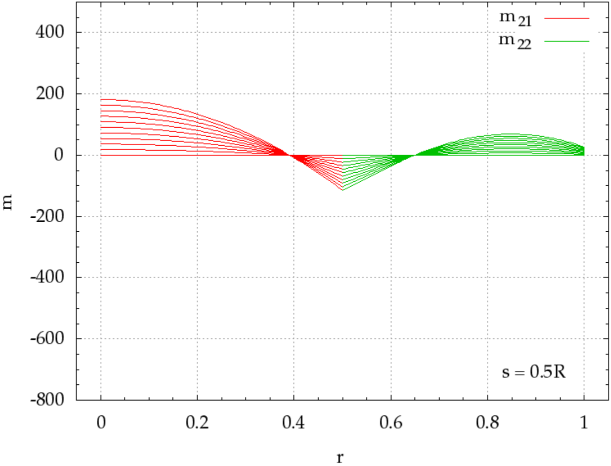

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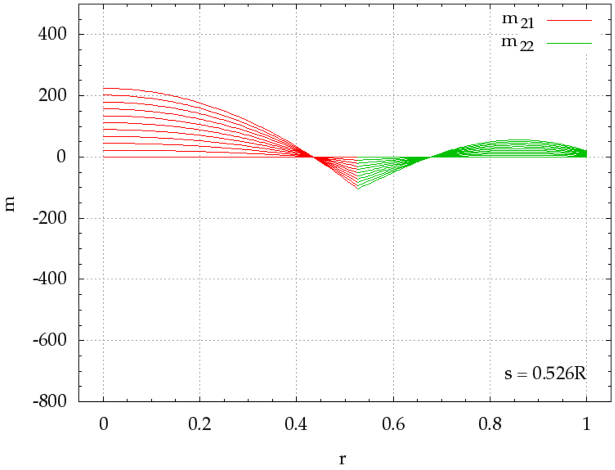

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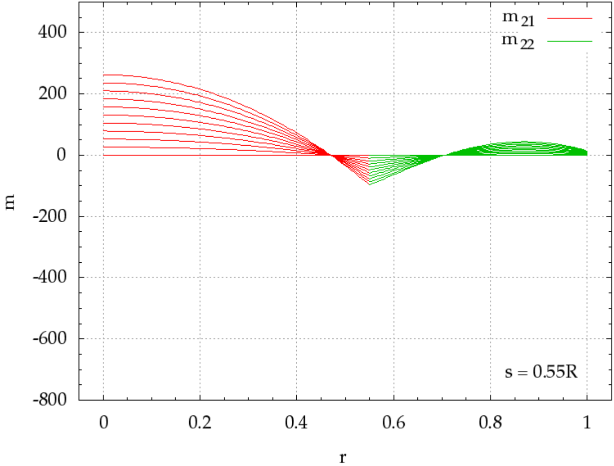

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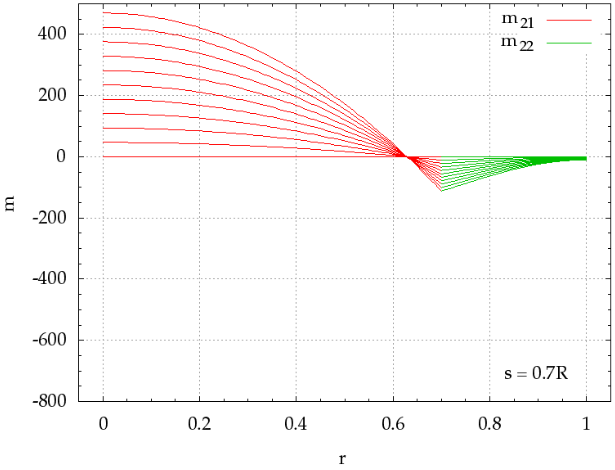

Concluding remarks

- For the optimal location of the additional support the deflection is always positive: $W(r) \geq 0$.
- The additional support enables to considerably reduce the maximal deflection of the plate.

The end

Thank You!