Elastic plastic bending of stepped annular plates

JAAN LELLEP
University of Tartu
Institute of Mathematics
J. Liivi Street 2, 50409 Tartu
ESTONIA
jaan.lellep@ut.ee

JULIA POLIKARPUS
University of Tartu
Institute of Mathematics
J. Liivi Street 2, 50409 Tartu
ESTONIA
julia.polikarpus@ut.ee

Abstract: Axisymmetric bending of annular plates subjected to the distributed transverse pressure is studied. The plates under consideration have piece wise constant thickness of carrying layers and are fully clamped at the inner edge whereas the outer edge is absolutely free. It is assumed taht the material of the plate is an ideal elastic plastic material obeying the square yield condition and the associated flow law in the stage of plastic deformations. Assuming that displacements, generalized stresses and strains remain axisymmetric and making use of the pure bending theory of thin plates the stress strain state of the plate is determine for the initial elastic and subsequent elastic plastic stages of the deformation. Numerical results are presented for a plate with a single step of the thickness.

Key–Words: annular plate, stepped thickness, elastic plastic material

1 Introduction

Plates and shells are widely used in various fields of technology and engineering. The behaviour of plates and shells in the range of elastic deformations has been studied by many authors (see Reddy, 2007; Vinson, 2005; Ventsel, Krauthammer, 2001).

It is wellknown that the structural material is used in more efficient manner and the ratio of the strength to weight is larger if inelastic deformations are taken into account when designing the structure. Although the early results of the behaviour of elastic plastic circular and annular plates have obtained a long ago by Hodge (1960); Tekinalp (1957) the most of the attention is paid to plates of constant thickness only. Comprehensive reviews of these investigations can be found in the books by Chakrabarty (2000), Kaliszky (1989), Save, Massonnet, Saxce (1997); Yu, Zhang (1996). Hodge introduced an essential simplification of inelastic problems in the case of a Tresca material making use of the yield surface consisting of two hexagons on the planes of moments and membrane forces, respectively. It was used by many investigators for getting approximate solutions. Among others, sherbourne, Srivastava (1971) found an analytical solution to the elastic plastic bending problem in the range of large deflections.

In the present paper the elastic plastic bending of annular plates of piece wise constant thickness is studied. It is assumed that the material of the plate clamped at the inner edge is an ideal plastic one obeying the square yield condition.

2 Problem formulation and basic hypothesis

Let us consider the axisymmetric bending of an annular plate subjected to the transverse pressure of intensity $P = P(r)$, where r is the current radius. Assume that the internal edge of the plate of radius a is clamped whereas the external edge of radius R is absolutely free.

The plate under consideration has sandwich-type cross section. It consists of two rims or carrying layers of thickness h whereas the space between the rims is stuffed with the core material. The latter is not able to resist to normal loads. Let the layer of the core material be of thickness $H = const$ and the carrying layers be of piece wise constant thickness, e. g.

$$h = h_j \tag{1}$$

for $r \in (a_j, a_{j+1})$, where $j = 0, 1, \ldots, n$. It is reasonable to denote $a_0 = a$ and $a_{n+1} = R$. The thicknesses $h_0, \ldots, h_n$ and the step location $a_1, \ldots, a_n$ are assumed to be given geometrical parameters of the plate.

The behaviour of the plate will be prescribed with the first order bending theory of thin plates corresponding to small deformations and small displacements (see Vinson, 2005; Reddy, 2007). The stress state is defined by bending moments M_1, M_2 in the radial and circumferential direction, respectively. Note that the membran forces will be neglected according to current approach. The third generalized stress component to be taken into account is the shear force Q but this component does not contribute to the strain energy when the bending theory is used. Moreover, the shear force, although appearing in the equilibrium conditions, can be eliminated from the equilibrium equations and thus does not involve in the set of governing equations.

As regards kinematical quantities the only displacement to be taken into account is the transverse deflection $W = W(r)$ whereas the radial displacement can be ignored according to the current approach. The strain components corresponding to the generalized stress components M_1, M_2 are the curvature components κ_1, κ_2 in the radial and circumferential direction, respectively.

It is assumed that the stress strain state induced by the axisymmetric transverse pressure is axisymmetric at each stage of the pressure. Thus the stress and strain components are defined at each point of the plate by the current radius and the given pressure level.

Material of the plate is assumed to be an ideal elastic plastic material obeying the square yield condition in the plastic (inelastic) stage of deformation.

The aim of the paper is to determine the transverse deflection as well as bending moments distributions in the elastic and subsequent inelastic stages of deformation for given transverse pressure levels.

3 Basic equations and concepts

Making use the theory of plates with small deflections and small deformations the equilibrium conditions of an element of an axisymmetric plate furnish the equations (see Reddy, 2007; Ventsel and Krauthammer, 2001)

$$\frac{d}{dr}(rM_1) - M_2 - rQ = 0, \frac{d}{dr}(rQ) = -Pr, \quad (2)$$

provided no external shear loading is applied to the plate. The assumptions of the classical thin plate theory require that transverse shear deformations be zero. However, the shear force Q is to be taken into account.

The strain components associated with the bending moments M_1, M_2 in the pure bending theory are

$$\kappa_1 = -\frac{d^2W}{dr^2}, \kappa_2 = -\frac{dW}{rdr}. \quad (3)$$

It is well known that in the case of lower values of the pressure loading the plate is pure elastic. The elastic behaviour of the material can be prescribed with Hooke's law. The latter is to be presented in the generalized form as (Reddy, 2007)

$$M_1 = D_j(\kappa_1 + \nu\kappa_2), M_2 = D_j(\kappa_2 + \nu\kappa_1) \quad (4)$$

where $j = 0, 1$ and in the case of sandwich plate

$$D_j = \frac{Eh_jH^2}{2(1-\nu^2)}. \quad (5)$$

In (4), (5) and henceforth E and ν denote the Young and Poisson modulus, respectively.

During the subsequent quasistatic increasing the external loading constitutive equations (4) hold good until the elastic limit is exhausted at an unknown point of the plate. In the case of the pressure of constant intensity the yield limit is reached at first at the center of the plate. After that the plate is subdivided into elastic and plastic regions, respectively. Let these regions be S_e and S_p, respectively. Since we are studying the plate of sandwich type and the carrying layers are thin no elastic-plastic state of deformatons occurs.

Assume that the material of the plate obeys the square yield condition and associated flow rule (joonis 2). Thus for $r \in S_p$ the stress state of the point is such that the point $(M_1(r),\ M_2(r))$ lies on a side of the square (joonis 2). It means that at each point of the plate are satisfied inequalities

$$|M_1| \leq M_{0j}, |M_2| \leq M_{0j} \quad (6)$$

where M_{0j} stands for the yield moment corresponding to the thickness h_j. It can be easily stated that (Chakrabarty, 2000)

$$M_{0j} = \sigma_0 h_j H, \quad (7)$$

σ_0 being the yield stress of the material. In an elastic region for $r \in S_e$ inequalities (6) are satisfied as strict inequalities.

Evidently, at the boundary of the plate requirements

$$M_1(R) = 0, Q(r) = 0 \quad (8)$$

and

$$W(a) = 0 \quad (9)$$

must be satisfied at each loading level.

Let us consider the governing equations separately in elastic and plastic regions, respectively. In elastic regions the stress strain state is determined according to (2) and (4). Substituting (4) and (3) in (2) easily leads to the equation

$$\frac{1}{r}\frac{d}{dr}\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dW}{dr}\right)\right]\} = \frac{P(r)}{D_j} \quad (10)$$

for $r \in (a_j, a_{j+1})$, provided $(a_j, a_{j+1}) \subset S_e$.

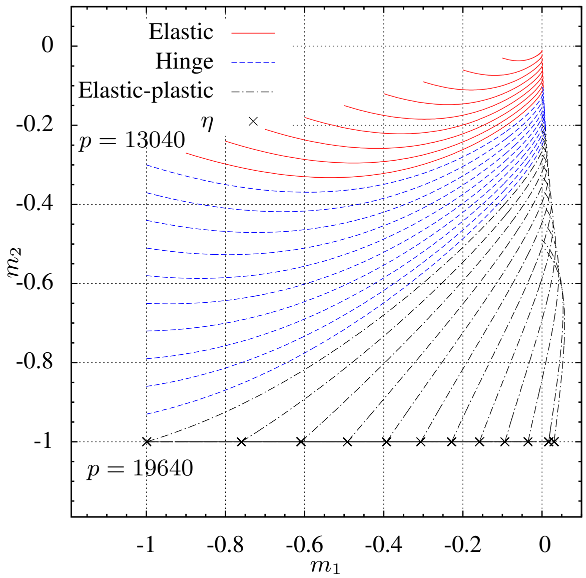

Figure 2: Moments m_1, m_2.

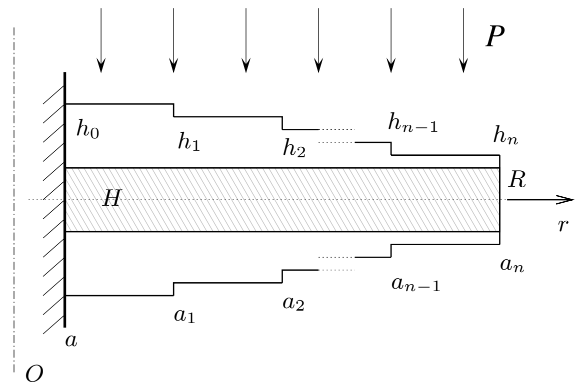

Figure 3: Cross-section.

In the following it is reasonable to use non-dimensional quantities

$$
\begin{aligned}
&\rho = \tfrac{r}{R}, m_1 = \tfrac{M_1}{M_*}, m_2 = \tfrac{M_2}{M_*}, q = \tfrac{RQ}{M_*}, \\
&\alpha = \tfrac{a}{R}, \alpha_j = \tfrac{a_j}{R}, p = \tfrac{PR^2}{M_*}, w = \tfrac{W}{H}, \qquad (11) \\
&\gamma_j = \tfrac{h_j}{h_*}, d_j = \tfrac{EH^2 h_j}{2(1-\nu^2)\sigma_0 R^2 h_*}
\end{aligned}
$$

where $M_* = \sigma_0 h_* H$ is the yield moment of a reference plate of constant thickness h_*.

Making use of variables (11) one can present the equilibrium equations (2) as

$$((\rho m_1)' - m_2)' + p\rho = 0 \qquad (12)$$

where prims denote the differentiation with respect to the non-dimensional radius ρ.

4 General solutions in elastic and plastic regions

Let us denote an elastic region (a_j, a_{j+1}) where the thickness of carrying layers is h_j by S_{ej}.

Making use of (10) and (11) it is easy to recheck that the general solution of (10) can be presented as

$$w = A_{1j}\rho^2 \ln\rho + A_{2j}\rho^2 + A_{3j}\ln\rho + A_{4j} + \frac{p\rho^4}{64 d_j} \qquad (13)$$

where $\rho \in S_{ej}$ and $d_j = \frac{D_j H}{M_* R^2}$. Arbitrary constants $A_{1j}, \dots A_{4j}$ will be determined from the boundary re-

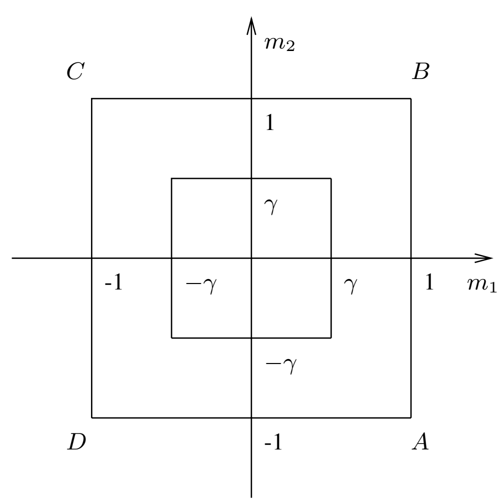

Figure 4: Square.

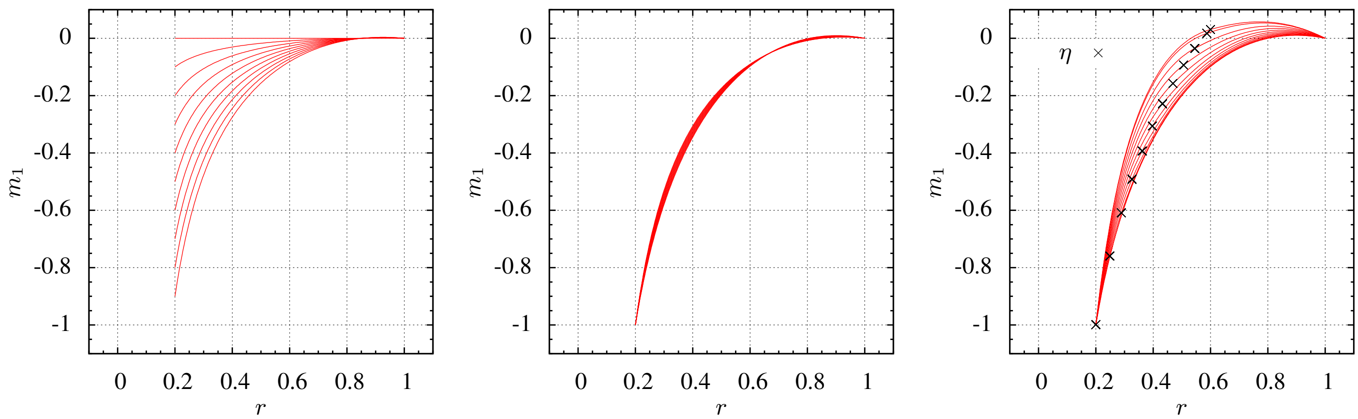

Figure 1: Moment m_1 on stages 1, 2, 3 from left to right.

quirements and continuity conditions for w, w', m_1 at the boundaries between elastic and plastic regions.

Non-dimensional bending moments can be determined according to (4) and (11) as

$$m_1 = -d_j\left(w'' + \frac{\nu}{\rho}w'\right), m_2 = -d_j\left(\frac{w'}{\rho} + \nu w''\right) \tag{14}$$

for $\rho \in S_{ej}$. The terms with derivatives w', w'' in (14) can be expressed as

$$w' = A_{1j}(2\rho\ln\rho + \rho) + 2A_{2j}\rho + \frac{A_{3j}}{\rho} + \frac{p\rho^3}{16d_j} \tag{15}$$

and

$$w'' = A_{1j}(2\ln\rho + 3) + 2A_{2j} - \frac{A_{3j}}{\rho^2} + \frac{3p\rho^2}{16d_j} \tag{16}$$

for $\rho \in S_{ej}$.

The third stress component besides bending moments is the shear force. It is reasonable to calculate it from the equilibrium equations (2) or (12). From (12), (2) one easily obtains the equation

$$(\rho q)' = -p\rho \tag{17}$$

which holds good over the entire plate. The solution of this equation which satisfies the boundary condition $q(1) = 0$ is

$$q = -\frac{p}{2}\left(\rho - \frac{1}{\rho}\right) \tag{18}$$

for $\rho \in (\alpha, 1)$.

The solution of basic equations in a plastic region S_p depends on the particular yield regime. It appears that in the present case the stress strain state in a plastic region of the plate corresponds to the sides AD or DC of the square yield condition (joonis 2). Let us consider these yield regimes in a greater detail.

In the case of the yield profile CD one has $m_1 = -\gamma_j$ for $\rho \in S_{pj} \subset (a_j, a_{j+1})$. However, it can be rechecked (see Chakrabarty, 2000; Save, Massonet, Saxce, 1997) that this regime can not take place at a region of finite length.

If the stress profile lies on the side AB of the yield square (joonis 2) then $m_2 = -\gamma_j$ and after integration of (12) with (18) one has

$$m1 = -\gamma_j - \frac{p}{2}\left(\frac{\rho^2}{3} - 1\right) + \frac{E_j}{\rho} \tag{19}$$

for $\rho \in S_{pj}$.

One of the constutive relations which has to be met in a plastic region is the gradientality or associated flow law (see Chakrabarty, 2000; Yu, Zhang, 1996). According to the associated flow law the vector of strain rate components must be normal to the yield surface at each point of a plastic region. In the present case it means that the vector $\dot{\vec{\kappa}} = (\dot{\kappa}_1, \dot{\kappa}_2)$ is parallel to the horizontal axis if the stress strain state corresponds to the side CD (joonis 2) and it is vertical if the yield regime AD takes place. Moreover, according to the Drucker's postulate the strain rate vector is to be directed along the outward normal to the yield surface at the current point. The strain rate vector can be determined by differentiating relations (3) with respect to time. Instead of time one can use the load intensity P. In the frameworks of the linear theory of plasticity corresponding relations can be integrated with respect to time and thus one can use the gradientality law replacing the strain rates with strain components.

For instance, in the case of the yield regime AD one has $\kappa_1 = 0$ and thus

$$w = A_j\rho + B_j \tag{20}$$

for $\rho \in S_{pj}$, where A_j, B_j are arbitrary constants. However, on the side CD of the yield condition (joonis 2) $\kappa_2 = 0$ and thus $w = const$. This is one of but not the only reason why the regime CD does not take place in a region of finite length.

5 The pure elastic stage of deformation

As the intensity of the pressure loading is increased from zero, the entire plate is elastic until the stress profile reaches a side of the yield condition. However, during the elastic stage the stress profile lies inside the square $ABCD$ (joonis 2) where $\gamma = \min \gamma_j$.

During this stage of laoding the deflection is defined by (13), bending moments and the shear force by (14) and (18), respectively. For determination of unknown constants one can use the boundary conditions $w'(a) = w(a) = 0$, $m_1(1) = 0$ and the continuity requirements

$$[w(\alpha_j)] = 0, [w'(\alpha_j)] = 0, [m_1(\alpha_j)] = 0 \qquad (21)$$

for $j = 1, \ldots, n$ where the square brackets denote finite jumps, e. g. $[z(\alpha_j)] = z(\alpha_j + 0) - z(\alpha_j - 0)$.

It appears that the general solution in the form (13) may involve inadequate solutions for particular cases. In order to avoid this one has to check if the shear force in the form (18) concides with that following from the first equation of the system (2).

Let us denote

$$\rho \bar{q} = (\rho m_1)' - m_2. \qquad (22)$$

Evidently, $\bar{q} = q$. Thus one has to check if the constraints

$$\bar{q}(\rho) = q(\rho) \qquad (23)$$

for $\rho \in S_{ej}$ ($j = 1, \ldots, n$) with the boundary condition $\bar{q}(1) = 0$ are satisfied. Making use of (13) - (16) and (18) with (22) it is easy to show that equalities (23) take place if

$$A_{1j} = -\frac{p}{8d_j} \qquad (24)$$

for $j = 0, \ldots, n$.

Thus, for determination of $3n + 3$ unknown constants A_{2j}, A_{3j}, A_{4j} ($j = 0, \ldots, n$) one has three boundary conditions and $3n$ continuity conditions (21).

Making use of (14), (15), (16) and taking (24) into account it is easy to determine the bending moments

$$m_1 = \tfrac{-p\rho^2(3+\nu)}{16} - \tfrac{p}{8}[2(1+\nu)\ln\rho + 3 + \nu] - $$
$$-2(1+\nu)A_{2j}d_j + \tfrac{A_{3j}d_j(1-\nu)}{\rho^2} \qquad (25)$$

and

$$m_2 = \tfrac{-p\rho^2(1+3\nu)}{16} - \tfrac{p}{8}[2(1+\nu)\ln\rho + 1 + 3\nu] - $$
$$-2(1+\nu)A_{2j}d_j + \tfrac{A_{3j}d_j(1-\nu)}{\rho^2} \qquad (26)$$

for $\rho \in (a_j, a_{j+1})$, $j = 0, \ldots, n$.

It is interesting to remark that the distribution of the shear force does not depend on the distribution of thicknesses, as it can be seen from (18). At the same time other stress components (bending moments) do depend on the thicknesses.

The elastic loading stage completes at the moment when the stress profile reaches the side CD of the yield square. In the case of a plate of constant thickness the plastic yielding happens at first at the internal edge of the plate for $\rho = \alpha$. At the limit stage between the fully elastic stage and inelastic stage for $p = p_0$

$$m_1(\alpha) = -\gamma_0. \qquad (27)$$

Note that, in principle, the plastic yielding may start elsewhere, as well. If, for instance, the inner annulus is very narrow and the thickness h_0 is large whereas the next annulus has very small thickness then the yield can start from the next annulus. However, these cases will not studied in the present paper.

6 Elastic plastic stage with the hinge circle

Assume that during this stage of deformation the plastic hinge circle is located at the internal edge of the plate for $\rho \neq \beta$ is elastic as during the previous stage. However, due to the hinge the boundary condition $w'(\alpha) = 0$ is no more valid. For determination of unknown constants A_{2j}, A_{3j}, A_{4j} one can use relations (21) - (23) with boundary conditions $w(\alpha) = 0$, $m_1(1) = 0$ and (27). Note that (24) remains valid, as well. The latter admits to present the deflection for $\rho \in [\alpha_j, \alpha_{j+1}]$ as

$$w = \frac{p\rho^2(\rho^2 - 8\ln\rho)}{64d_j} + A_{2j}\rho^2 + A_{3j}\ln\rho + A_{4j}. \qquad (28)$$

Making use of (28), (25) and satisfying boundary conditions $w(\alpha) = 0$, $m_1(\alpha) = -\gamma_0$ results in

$$A_{20}\alpha^2 + A_{30}\ln\alpha = -A_{40} - \tfrac{p\alpha^2(\alpha^2 - 8\ln\alpha)}{64d_0},$$
$$-2A_{20}(1+\nu)d_0 + A_{30}\tfrac{d_0(1-\nu)}{\alpha^2} = \qquad (29)$$
$$= -\gamma_0 + \tfrac{p\alpha^2(3+\nu)}{16} + \tfrac{p[2(1+\nu)\ln\alpha + 3 + \nu]}{8}$$

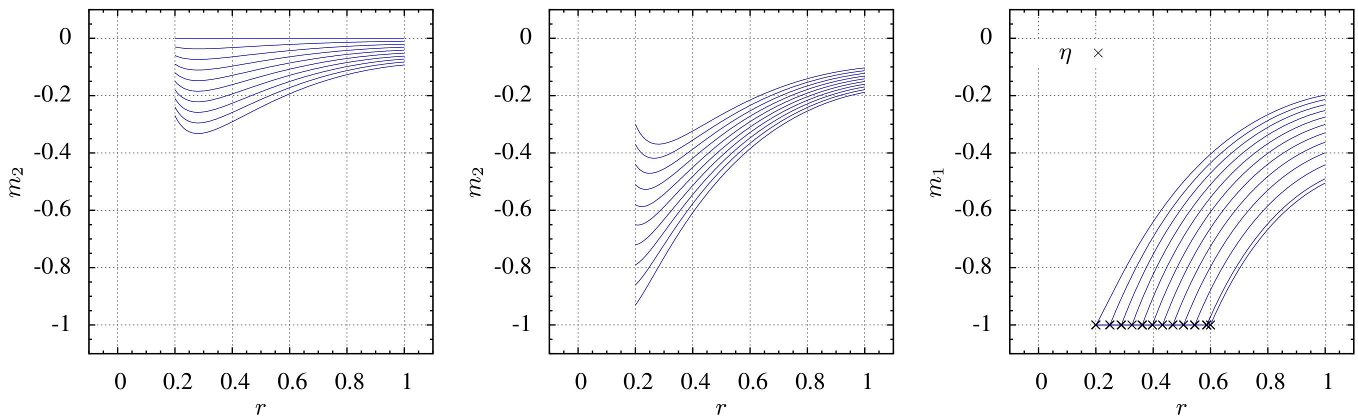

Figure 5: Moment m_2 on stages 1, 2, 3 from left to right.

Finally, employing the continuity conditions (21) with boundary conditions (29) and $m_1(1) = 0$ admits to determine unknown constants A_{2j}, A_{3j}, A_{4j} for each $j = 0, \ldots, n$.

This stage of determination will be completed when the stress profile at $\rho = \alpha$ reaches the point D (joonis 2) so that $m_2(\alpha) = -\gamma_0$. Let the corresponding value of the external load intensity be p_1.

7 The elastic plastic stage with a plastic region of finite length

It is reasonabelt to assume that during the subsequent increasing the tansverse pressure plastic deformations take place for $\rho \in S_{p0}$, $\rho \in [\alpha, \eta]$ where η is a previously unknown constant. The plastic region corresponds to the yield regime DA (joonis 2). Thus, for $\rho \in (\alpha, \eta)$

$$m_2 = -\gamma_0. \tag{30}$$

The distribution of the radial bending moment m_1 can be calculated by (19) taking $j = 0$. As at $\rho = \beta$, $m_1 = -\gamma_0$ the arbitrary constant E_0 is to be

$$E_0 = \frac{\alpha p}{2}\left(\frac{\alpha^2}{3} - 1\right). \tag{31}$$

Thus the bending monet is defined as

$$m_1 = -\gamma_0 - \frac{p}{2}\left(\frac{\rho^2}{3} - 1\right) + \frac{\alpha p}{2\rho}\left(\frac{\alpha^2}{3} - 1\right) \tag{32}$$

for $\rho \in [\alpha, \eta]$.

The tranverse deflection has the form (20) in the plastic region S_{p0}. According to the boundary condition $w(\alpha) = 0$ must be $B_0 = -A_0\alpha$. Thus

$$w = A_0(\rho - \alpha) \tag{33}$$

for $\rho \in [\alpha, \eta]$.

Since we can now define $w(\eta)$, $w'(\eta)$, $m_1(\eta)$ the subsequent solution procedure is similar to that accomplished in the previous section. Note that for $\rho > \eta$ the plate is elastic. For determination of the parameter η one has to use the continuity of the moment m_2 taking $m_2(\eta) = -\gamma_0$.

8 Several plastic regions

The previous stage of loading terminates at the moment when plastic yielding takes place in the section S_{p1} or in another section. Let us assume for the conceteness sake that the stress profile reaches to the corresponding yield level at $\rho = \alpha_1$ when $p = p_2$. Thus for $p \geq p_2$ the plastic deformations take place in the region (α_1, η_1) as well as in (α, η) which continues the extension. It means that $m_2 = -\gamma_0$ for $\rho \in (\alpha_1, \eta_1)$.

The bending moment distribution for $\rho \in S_{p1}$ can be defined according to (19) with the unknown constant E_1. Similarily the deflection w is given by (20) with unknown constants A_1, B_1.

The procedure of the determination of constants of integration is similar to that accomplished in the previous case. Now we have to take into account that the region (η, α_1) between plastic regions remain elastic. Here we have unknown constants A_{21}, A_{31}, A_{41}. The number of unknowns in each region is, thus, three. For determination of these constants the continuity requirements for m_1, w, w' are applicable. Finally, the parameters η, η_1 are to be determined from equations $m_2(\eta) = -\gamma_0$ and $m_2(\eta_1) = -\gamma_1$ where m_2 is calculated for an elastic region according to (26) with previously defined constants A_{2j}, A_{3j}.

9 Numerical results

Results of calculations in the case of plates with a single step are presented in Fig. 3. The results regard to the plate with inner radius $a = 0.2R$.

The stress profiles on the plane of moments m_1, m_2 are shown in Fig. 3 for different values on the load intensity. It can be seen from Fig. 3 that the profiles corresponding to smaller values of the load p lie wholly inside the square $|m_1| \leq 1$, $|m_2| \leq 1$. When the load intensity increases until $p = p_1$ the end of the profile reaches the side $m_1 = -1$ and for $p = p_2$ the corner point where $m_1 = m_2 = -1$. During subsequent growth of the load intensity the end of the stress profile lies on the side $m_2 = -1$ as it was expected theoretically.

Distributions of bending moments m_1 and m_2 are presented in Fig. 4 and Fig. 5, respectively. it can be seen that when the load increases the stress state tends to the pure plastic state. In the case of a plate of constant thickness in the pure plastic state $m_2 \equiv -1$. In the case of a stepped plate it can be such that $m_2 = -\gamma_j$ for $\rho \in (\alpha_j, \alpha_{j+1})$, $j = 0, \ldots, n$.. However, the question which is the stress state at the plastic collapse can be answered by the limit analysis of the plate of particular shape.

10 Concluding remarks

A method for theoretical investigation of axisymmetric plates subjected to the distributed transverse pressure was developed. The material of plates was assumed to be an ideal elastic plastic material obeying the square yield condition and the associated flow law in the range of inelastic deformations. In order to get maximal simplicity of the posed problem hardening of the material as well as geometrical non-linearity of the plate behaviour were neglected.

It was assumed that the plates under consideration had piece wise constant thickness with arbitrary number of steps. Exact solutions were developed for the case when the plate is clamped at the inner edge whereas the outer edge is absolutely free. As a result of the solution procedure a succession of stress states which are in equilibrium with the external loading were constructed that led from the wholly elastic to the elastic plastic state and finally to the plastic collapse state. Since a plate of variable thickness can be approximated with an appropriate choise of the piece wise constant thickness the present solution technique is applicable for approximate solution of similar problems in the cases of plates of variable thickness.

Acknowledgements: The partial support from the Estonian Science Foundation Grant ETF 7461 and from the Target-Financed Project SF 0180081908 is gratefully acknowledged.

References:

[1] J. Chakrabarty, *Applied Plasticity,* Berlin, Springer, 2000.

[2] F. W. French, Elastic-plastic analysis of centrally clamped annular plates under uniform loads, *Journal of the Franklin Institute,* 277, 6, 1964, pp. 575–592.

[3] M. Gorji, A. R. Akileh, Elastic-plastic bending of annular plates with large deflections, *Computers and Sructures,* 34, 4, 1990, pp. 537–548.

[4] P. G. Hodge, *Plastic Analysis of Structures,* New-York, Krieger, 1981.

[5] S. Kaliszky, *Plasticity. Theory and Engineering Applications,* Amsterdam, Elsevier, 1989.

[6] J. Lellep, A. Mürk, Optimization of inelastic annular plates with cracks, *Struct. Multidiscipl. Optim.,* 35, 1, 2008, pp. 1–10.

[7] J. Lellep, J. Polikarpus, Optimization of elastic plastic circular plates under axisymmetric loading, *Proc. 20th Intern. Conference "Continuous Optimization and Knowledge-Based Technologies". Ed. L. Sakalauskas, G. W. Weber, E. K. Zavadskas,*, Vilnius, Technika, 2008, pp. 291–295.

[8] J. N. Reddy, *Theory and Analysis of Elastic Plates and Shells,* Boca Raton, CRC Press, 2007.

[9] M. A. Save, C. E. Massonnet, G. De Saxce, *Plastic Limit Analysis of Plates, Shells and Disks,* Amsterdam, Elsevier, 1997.

[10] A. N. Sherboure, A. Srivastava, Elastic-plastic bending of restrained pin-ended circular plates, *Int. J. Mech. Sci.,* 13, 1971, pp. 231–241.

[11] B. Tekinalp, Elastic-plastic bending of a built-in circular plates under a uniformly distributed load, *J. Mech. Phys. Solids,* 5, 2, 1957, pp. 135–142.

[12] M. Upadastra, J. Peddieson, G. R. Buchanan, Elastic compensation simulation of elastic-plastic axisymmetric circular plate bending using a deformation model, *Int. J. Non-Linear Mech.,* 41, 2006, pp. 377–387.

[13] E. Ventsel, T. Krauthammer, *Thin Plates and Shells: Theory, Analysis and Applications,* New-York, Marcel Dekker, 2001.

[14] J. Vinson, *Plate and Panel Structures of Isotropic, Composite and Piezoelectric Materials, Including Sandwich Construction,* Dordrecht, Springer, 2005.

[15] T. X. Yu, L. C. Zhang, *Plastic Bending. Theory and Applications,* Singapore, World Scientific, 1996.