Abstract

An optimization technique is developed for circular plates of piece wise constant thickness. The plates under consideration have been manufactured of an ideal elastic plastic material obeying Tresca's yield criterion. Necessary optimality conditions are derived with the aid of the theory of optimal control. Obtained system of differential-algebraic equations is solved numerically in the case of the plate with single step of the thickness. Effectiveness of the design is assessed numerically.

Keywords: circular plates, optimization, minimum weight, elastic-plastic material, Tresca's condition

Optimization of Elastic-plastic Circular Plates

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1 Introduction

Thin-walled plates and shells are important structural elements. These elements are widely used in the civil engineering, aeronautics and in various fields of technology. In the available literature a lot of attention is paid to optimization of plates and shells made of either pure elastic or ideal inelastic materials (Cherkaev, 2000; Kaliszky, 1989; Lellep and Mürk, 2008; Zyczkowski, 1992). However, there are only a few papers dedicated to optimization of structural elements made of elastic-plastic materials. On the other hand, elastic plastic analysis of circular and annular plates has deserved the attention of investigators during many years (Hodge, 1981; Kaliszky, 1989; Chakrabarty, 2000; Yu, Zhang, 1996).

It is intresting to remark that in the earlier investigations plates made of a von Mises material are studied (Sokolovsky, 1969; Eason, 1961; Popov et al, 1967). Later various numerical methods are suggested by Gorji and Akileh (1990), Ohasi and Murakami (1967).

Bending of elastic plastic axisymmetric plates made of a Tresca material was studied by Hodge (1981), Haythornthwaite (1954). Calladine (1968) suggested a simplified method for large deflection analysis in the case of an elastic plastic material. Ehakrabarty (1968) presented an approximate solution for a clamped circular plate deformed by a concentrated load acting at the center whereas Mazumdar and Jain (1989) developed a solution procedure for plates of arbitrary shape.

Recently Upadastra et al (2006) suggested the elastic compensation method for investigation of elastic plastic circular and annular plates subjected to axisymmetric loadings.

Kaliszky and Logo (2002, 2006) presented a layout and shape optimization method for optimization of elastic-plastic disks in the presence of constraints on displacements and deformations. In (Kaliszky and Logo, 2003, 2006) methods for optimal design of elastic plastic structures subjected to short time dynamic pressure or impact loading were suggested. Lepik (1994) considered elastic plastic stepped beams under distributed dynamic pressure. In (Lepik, 1995)the optimal layout of internal supports to a beam under quasi-static transverse loading was established.

In the present paper we develop an optimal project for a stepped circular plate. The plate is made of an ideal elastic plastic Tresca material and loaded with the uniformly distributed transverse pressure. Necessary optimality conditions are derived with the aid of the theory of optimal control (Bryson and Ho, 1975; Hocking, 2001; Hull, 2003).

2 Formulation of the problem

Consider a circular plate of radius R subjected to the uniformly distributed transverse pressure of intensity P. Let the plate be simply supported at the edge. We are treating the plate as a structure consisting of two carrying metallic layers and of a core material between the rims. The thickness of the core is constant over the plate and equals to H. However the thickness of carrying layers h is piece wise constant whereas

$$h = h_j, \quad r \in [a_j, a_{j+1}]; \quad j = 0, \dots, n.$$

Here h_j ($j = 0, \dots, n$) and a_i ($i = 1, \dots, n$) stand for unknown constant and r is the current radius. The system of polar coordinates with the origin O located at the centre of the plate is used. However, the strain-stress state of the plate is assumed to be symmetric. Thus the polar angle Θ is unnecessary in the present case. the number of steps n is assumed to be fixed preliminarity.

The behaviour of the plate structure under transverse loading is modelled as ideal elastic-plastic bending. Work hardening of the material will be neglected.

The aim of this paper is to establish the minimum weight design of the plate in the range of elastic plastic deformations for constrained deflections. when minimising the weight of the plate, governing equations of the plate theory will be taken into account.

Evidently, instead of the weight one can minimize the material volume of carrying layers. The volume of carrying layers can be expressed as

$$V = \pi[h_0 a_1^2 + \sum_{j=1}^{n} h_j(a_{j+1}^2 - a_j^2)], \tag{1}$$

provided $a_{n+1} = R$.

3 Governing equations

In the bending theory of thin plates internal stress state at each point of the plate is defined by bending moments M_1 and M_2 in radial and circumferential directions respectively. According to equilibrium conditions of a plate element bending moments have to satisfy equilibrium equation (Chakrabarty, 2000; Reddy, 2007)

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r}(rM_1) - M_2 = rQ, \\ \dfrac{\mathrm{d}}{\mathrm{d}r}(rQ) = -Pr. \end{cases} \tag{2}$$

Geometrical relations have the form

$$\begin{aligned} \kappa_1 &= -\frac{\mathrm{d}^2 W}{\mathrm{d}r^2}, \\ \kappa_2 &= -\frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}, \end{aligned} \tag{3}$$

where W stands for the tranverse deflection and κ_1, κ_2 are components of the curvature.

In an elastic region Hooke's law holds well. According to Hooke's law (Reddy, 2007; Ventsel and Krauthammer, 2001)

$$\begin{aligned} M_1 &= D_j(\kappa_1 + \nu\kappa_2), \\ M_2 &= D_j(\kappa_2 + \nu\kappa_1). \end{aligned} \tag{4}$$

Here E is the Young modulus, ν stands for the Poisson's modulus and

$$D_j = \frac{E h_j H^2}{2(1 - \nu^2)} \tag{5}$$

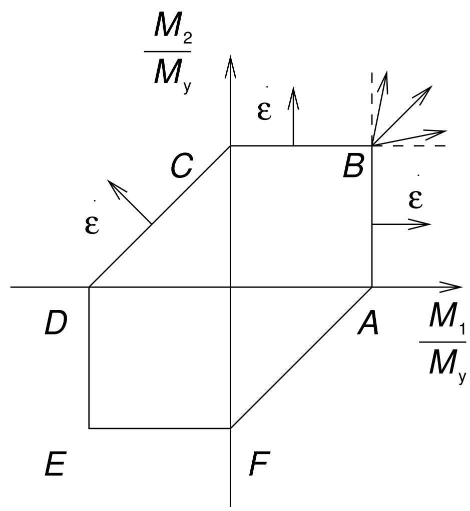

Figure 1: Tresca's yield hexagon.

in the case of a sandwich plate.

It is assumed that the behaviour of the material in plastic stage corresponds to Tresca's yield condition (Fig. 1) and to associated gradientality law (Chakrabarty, 2000; Kaliszky, 1989). The latter means that if the stress point is lying at an edge of Tresca's hexagon then the vector of curvatures with components (3) is directed towards external normal to the edge. It will be shown that in the case of a circular plate subjected to undirectional transverse loading the yield regime $M_2 = M_0$ takes place, where M_0 stands for the yield moment. It is well known that $M_0 = \sigma_0 hH$, σ_0 being the yield stress of the plate material.

If the stress state of the plate corresponds to an internal point of the side BC of the yield hexagon, then according to the gradientality law $\dot{\kappa}_1 = 0$, $\dot{\kappa}_2 \geq 0$. Thus

$$\frac{\mathrm{d}^2\dot{W}}{\mathrm{d}r^2} = 0$$

in this case.

If the stress strain state of the plate in certain region corresponds to a non-regular point of the yield curve (for instance, the corner point B in Fig. 1), then the strain rate vector lies inside the angle formed by external normals to crossing sides.

Note that if the stress strain state corresponds to an interiour point of the yield hexagon (Fig. 1) then the plate material remains elastic in this region and Hooke's law holds good.

Although being of piecewise constant thickness the carrying layers are relatively thin. Therefore stress distributions across the thickness can be considered as constants and the stress state can be either elastic or pure plastic; no elastic-plastic state occurs.

It is reasonable to introduce following non-dimensional quantities

$$\begin{aligned} &\rho = \frac{r}{R}, \quad m_1 = \frac{M_1}{M_*}, \quad m_2 = \frac{M_2}{M_*}, \quad k = \frac{M_* R^2}{HD_*}, \\ &\alpha = \frac{a_j}{R}, \quad p = \frac{PR^2}{6M_*}, \quad w = \frac{W}{H}, \quad \gamma_j = \frac{h_j}{h_*}, \end{aligned} \tag{6}$$

where h_* is the thickness of carrying layers of the reference plate. In the present study we take $h_* = h_0$ whereas $M_* = \sigma_0 h_* H$. The parameters a_j stand for the radii of steps.

Making use of (2) - (6) one can present the set of governing equations in the elastic region as

$$\begin{aligned} w' &= z, \\ z' &= -\frac{km_1}{\gamma_j} - \frac{\nu}{\rho} z, \\ m_1' &= \frac{(\nu^2 - 1)\gamma_j}{k} \frac{z}{\rho^2} - \frac{m_1}{\rho}(1 - \nu) - p\rho, \end{aligned} \tag{7}$$

for $\rho \in (\alpha_j, \alpha_{j+1})$; $j = 1, \ldots, n$. In the plastic region according to the flow law $w'' = 0$. Moreover, at each point of the central region equilibrium equations must be satisfied. Thus

$$\begin{aligned} w' &= z, \\ z' &= 0, \\ m_1' &= \frac{1}{\rho}(\gamma_j - m_1) - 3p\rho \end{aligned} \tag{8}$$

hold good in a plastic region for $\rho \in (\alpha_j, \alpha_{j+1})$.

Let us consider the case when $\eta \leq \alpha_1$ in a greater detail. Integrating the system (8) yields

$$\begin{aligned} w &= A\rho + w_0, \\ z &= A \end{aligned} \tag{9}$$

and

$$m_1 = \gamma_0 - p\rho^2, \tag{10}$$

where the boundary conditions

$$\begin{aligned} w(0) &= w_0, \\ m_1(0) &= m_2(0) = \gamma_0 \end{aligned} \tag{11}$$

have been taken into account. Thus, for $\eta \leq \alpha_1$ one has

$$w(\eta) = A\eta + w_0 \tag{12}$$

and

$$m_1(\eta) = \gamma_0 - p\eta^2. \tag{13}$$

If, however, $\eta > \alpha_1$ then it follows from (8) and (11) that variables w and z satisfy (9), as previously. However, for determination of the bending moment m_1 one has to use equations

$$m_1' = \begin{cases} \frac{1}{\rho}(\gamma_0 - m_1) - 3p\rho, & \rho \in (0, \alpha_1), \\ \frac{1}{\rho}(\gamma_1 - m_1) - 3p\rho, & \rho \in (\alpha_1, \eta). \end{cases} \tag{14}$$

Integrating these equations and satisfying the continuity condition $m_1(\alpha-) = m_1(\alpha+)$ with (11) one obtains

$$m_1 = \begin{cases} \gamma_0 - p\rho^2, & \rho \in (0, \alpha_1), \\ \gamma_1 + (\gamma_0 - \gamma_1)\alpha_1 - p\rho^2, & \rho \in (\alpha_1, \eta). \end{cases} \tag{15}$$

Thus, according to (15)

$$m_1(\eta) = \gamma_1 + (\gamma_0 - \gamma_1)\alpha_1 - p\eta^2 \tag{16}$$

in this case.

One has to take into account that A and η in (9) - (16) stand for unknown parameters.

The problem posed above will be considered as an optimal control problem with state equations (7), (8) and the cost function (1). Assuming that $\alpha_0 = \eta$ one can present the performance index as

$$J = \gamma_0 \eta^2 + \sum_{j=0}^{n} \gamma_j (\alpha_{j+1}^2 - \alpha_j^2). \tag{17}$$

The edge of the plate is simply supported. Therefore

$$w(1) = 0, \quad m_1(1) = 0. \tag{18}$$

4 Necessary optimality conditions

For the sake of simlicity let us consider the case where $n = 1$ and $\eta \leq \alpha$ (in this case indexes may be omitted and $\alpha_1 = \alpha$).

In order to minimise (17), where $n = 1$ under constaints (7) - (16) let us introduce an exended functional (Hull, 2005; Lellep and Mürk, 2008; Bryson and Ho, 1975)

$$\begin{aligned} J_* = {} & \gamma_0 \alpha^2 + \gamma_1 (1 - \alpha^2) + \int_\eta^\alpha \left\{ \psi_1 (w' - z) + \psi_2 \left(z' + \frac{k m_1}{\gamma_0} + \frac{\nu}{\rho} z \right) + \right. \\ & \left. + \psi_3 \left[m_1' - \gamma_0 (\nu^2 - 1) \frac{z}{k\rho^2} + \frac{m_1}{\rho} (1 - \nu) + p\rho \right] \right\} \mathrm{d}\rho + \\ & + \int_\alpha^1 \left\{ \psi_1 (w' - z) + \psi_2 \left(z' + \frac{k m_1}{\gamma_1} + \frac{\nu}{\rho} z \right) + \right. \\ & \left. + \psi_3 \left[m_1' - \gamma_1 (\nu^2 - 1) \frac{z}{k\rho^2} + \frac{m_1}{\rho} (1 - \nu) + p\rho \right] \right\} textd\rho + \\ & + \lambda_1 [w(\eta) - A\eta - w_0)] + \lambda_2 [z(\eta) - A] + \lambda_3 [m_1(\eta) - \gamma_0 + p\eta^2]. \end{aligned} \tag{19}$$

In (19) ψ_1, ψ_2, ψ_3 stand for adjoint variables and λ_1, λ_2, λ_3 are unknown Lagrange' multipliers.

Necessary optimality conditions of the functional (19) can be presented as $\Delta J_* = 0$ where ΔJ_* is the total variation of the functional J_*. When calculating total variations one has to follow the samples (see Lellep, 1991; Lellep, Puman, 2007; Lellep, Mürk, 2008; Lellep, Tungel, 2005)

$$\begin{aligned} & \Delta \int_\eta^\alpha F \mathrm{d}\rho = \delta \int_\eta^\alpha F \mathrm{d}\rho + F|_{\rho=\alpha} \cdot \Delta\alpha - F|_{\rho=\eta} \cdot \Delta\eta, \\ & \Delta \int_\alpha^1 F \mathrm{d}\rho = \delta \int_\alpha^1 F \mathrm{d}\rho - F|_{\rho=\alpha} \cdot \Delta\alpha \end{aligned} \tag{20}$$

and

$$\begin{aligned} & \Delta y(\alpha \pm 0) = \delta y(\alpha \pm 0) + y'(\alpha \pm 0) \cdot \alpha, \\ & \Delta y(\eta) = \delta y(\eta) + y'(\eta) \cdot \Delta\eta \end{aligned} \tag{21}$$

where

$$y(\alpha \pm 0) = \lim_{\rho \to \alpha \pm 0} y(\rho).$$

Here y stands for a state variable (in the present case the state variables are w, z and m_1) and δy is the weak variation of the variable y.

Calculating the total variation of (19) and integrating by parts terms $\psi_1\delta w'$, $\psi_2\delta z'$ and $\psi_3\delta m_1'$ one obtains

$$
\begin{aligned}
&\Delta J_* = \alpha^2\Delta\gamma_0 + 2\gamma_0\alpha\Delta\alpha + (1-\alpha^2)\Delta\gamma_1 - 2\alpha\gamma_1\Delta\alpha+\\
&+\int_\eta^\alpha \Bigg\{ -\psi_1'\delta w - \psi_1\delta z - \psi_2'\delta z+\\
&+\psi_2\left(\frac{k}{\gamma_0}\delta m_1 - \frac{km_1}{\gamma_0^2}\Delta\gamma_0 + \frac{\nu}{\rho}\delta z\right) - \psi_3'\delta m_1+\\
&+\psi_3\left[\frac{(1-\nu^2)\gamma_0}{k\rho^2}\delta z + \frac{(1-\nu^2)z}{k\rho^2}\Delta\gamma_0 + \frac{1-\nu}{\rho}\delta m_1 + \rho\delta p\right]\Bigg\}\,\mathrm{d}\rho+\\
&+\int_\alpha^1 \Bigg\{ -\psi_1\delta w - \psi_1\delta z - \psi_2'\delta z+\\
&+\psi_2\left(\frac{k}{\gamma_1}\delta m_1 - \frac{km_1}{\gamma_1^2}\Delta\gamma_1 + \frac{\nu}{\rho}\delta z\right) - \psi_3'\delta m_1+\\
&+\psi_3\left[\frac{(1-\nu^2)\gamma_1}{k\rho^2}\delta z + \frac{(1-\nu^2)z}{k\rho^2}\Delta\gamma_1 + \frac{1-\nu}{\rho}\delta m_1 + \rho\delta p\right]\Bigg\}\,\mathrm{d}\rho+\\
&+(\psi_1\delta w + \psi_2\delta z + \psi_3\delta m_1)\,\Big|_\eta^{\alpha-} + (\psi_1\delta w + \psi_2\delta z + \psi_3\delta m_1)\Big|_{\alpha+}^1+\\
&+\lambda_1[\Delta w(\eta) - A\Delta\eta - \eta\Delta A] + \lambda_2[\Delta z(\eta) - \Delta A]+\\
&+\lambda_3[\Delta m_1(\eta) - \Delta\gamma_0 + 2p\eta\Delta\eta] = 0.
\end{aligned}
\tag{22}
$$

Since δw, δz, δm_1 are arbitrary variations of state variables, it immediately follows from the equation (22) that the adjoint equations

$$
\begin{aligned}
\psi_1' &= 0,\\
\psi_2' &= -\psi_1 + \frac{\nu}{\rho}\psi_2 + \frac{(1-\nu^2)\gamma_j}{k\rho^2}\psi_3,\\
\psi_3' &= \frac{k}{\gamma_j}\psi_2 + \frac{1-\nu}{\rho}\psi_3,
\end{aligned}
\tag{23}
$$

hold well for $\rho \in S_j$, where $j = 0, 1$. Here S_0, S_1 stand for regions $[\eta, \alpha]$ and $[\alpha, 1]$ respectively.

Similarly one obtains from (22) that

$$
\alpha^2 + \int_\eta^\alpha \left[-\frac{k}{\gamma_0^2}\psi_2 m_1 + \frac{\psi_3 z}{k\rho^2}(1-\nu^2)\right]\mathrm{d}\rho - \lambda_3 = 0 \tag{24}
$$

and

$$
1-\alpha^2 + \int_\alpha^1 \left[-\frac{k}{\gamma_1^2}\psi_2 m_1 + \frac{\psi_3 z}{k\rho^2}(1-\nu^2)\right]\mathrm{d}\rho = 0 \tag{25}
$$

Note that the state variables w, z, m_1 are continuous at each $\rho in[\eta, 1]$. Thus the total variations of state variables must be continuous, as well. Therefore,

$$
\begin{aligned}
&\Delta w(\alpha-) = \Delta w(\alpha+) = \Delta w(\alpha),\\
&\Delta z(\alpha-) = \Delta z(\alpha+) = \Delta z(\alpha),\\
&\Delta m_1(\alpha-) = \Delta m_1(\alpha+) = \Delta m_1(\alpha).
\end{aligned}
$$

Substituting the variations $\delta w(\alpha \pm 0)$, $\delta z(\alpha \pm 0)$, $\delta m_1(\alpha \pm 0)$ in (22) and taking (23) - (25) into account leads to the equation

$$\begin{aligned}
&(2\alpha\gamma_0 - 2\alpha\gamma_1)\Delta\alpha + \psi_1(\alpha-)(\Delta w(\alpha) - w'(\alpha-)\Delta\alpha)+\\
&+\psi_2(\alpha-)(\Delta z(\alpha) - z'(\alpha-)\Delta\alpha)+\\
&+\psi_3(\alpha-)(\Delta m_1(\alpha) - m_1'(\alpha-)\Delta\alpha) - \psi_1(\alpha+)(\Delta w(\alpha) - w'(\alpha+)\Delta(\alpha))-\\
&-\psi_2(\alpha+)(\Delta z(\alpha) - z'(\alpha+)\Delta(\alpha)) - \psi_3(\alpha+)(\Delta m_1(\alpha) - m_1'(\alpha+)\Delta\alpha)-\\
&-\psi_1(\eta)(\Delta w(\eta) - w'(\eta)\Delta(\eta)) - \psi_2(\eta)(\Delta z(\eta) - z'(\eta)\Delta\eta))-\\
&-\psi_3(\eta)(\Delta m_1(\eta) - m_1'(\eta)\Delta\eta) + \psi_1(1)\Delta w(1) + \psi_2(1)\Delta z(1) + \psi_3(1)\Delta m_1(1)+\\
&+\lambda_1(\Delta w(\eta) - A\Delta\eta - \eta\Delta A) + \lambda_2(\Delta z(\eta) - \Delta A)+\\
&+\lambda_3(\Delta m_1(\eta) - \Delta\gamma_0 + 2p\eta\Delta\eta) = 0.
\end{aligned} \tag{26}$$

In (26) the total variations of state variables $\Delta w(\eta)$, $\Delta z(\eta)$, $\Delta m_1(\eta)$ as well as $\Delta z(1)$, can be considered as independent variations (see Bryson, Ho, 1975; Hull, 2003). Note that due to the boundary conditions $m_1(1) = 0$, $w(1) = 0$ the variations $\Delta m_1(1)$ and $\Delta w(1)$ vanish. Arbritrary are also the variations $\Delta w(\alpha)$, $\Delta z(\alpha)$, $\Delta m_1(\alpha)$. Thus one easily obtains from (26) the tranversality condition

$$\psi_2(1) = 0, \tag{27}$$

continuity conditions of adjoint variables

$$\psi_i(\alpha - 0) = \psi_i(\alpha + 0) \tag{28}$$

for each $i = 1, 2, 3$ and

$$\psi_i(\eta) = \lambda_i. \tag{29}$$

Finally, due to the independence of ΔA and $\Delta\eta$ one obtains

$$\lambda_1\eta + \lambda_2 = 0 \tag{30}$$

and

$$[2\lambda_3 p\eta + \psi_2(\eta)z'(\eta+) + \psi_3(\eta)m_1'(\eta)] \mid_{\eta+0} = 0. \tag{31}$$

Finally, substituting (27) - (31) in (26) yields

$$\begin{aligned}
&2(\gamma_0 - \gamma_1)\alpha - \psi_2(\alpha)z'(\alpha-) - \psi_3(\alpha)m_1'(\alpha-)+\\
&+\psi_2(\alpha)z'(\alpha+) + \psi_3(\alpha)m_1'(\alpha+) = 0.
\end{aligned} \tag{32}$$

Note that when deriving (32) continuity of adjoint variables (28) has been taken into account.

5 Optimal design of the plate

In order to determine optimal parameters of the stepped plate one has to integrate state equations (7), (8) and adjoint equations (23) while satisfying appropriate boundary conditions. Subsequently one has to define the design parameters and Lagrange' multipliers from the system consisting of equations (24) - (32).

In order to solve the state equations (7) let us express from the third equation in (7)

$$z = \frac{k}{(\nu^2 - 1)\gamma_j}(\rho^2 m_1' + \rho m_1(1 - \nu) + p\rho^3). \tag{33}$$

Differentiating (33) with respect to ρ and making use of (7) yields the equation

$$m_1'' + \frac{3}{\rho} m_1' = -p(3+\nu). \tag{34}$$

Evidently, the general solution of (34) can be presented as

$$m_1 = -\frac{p}{8}(3+\nu)\rho^2 - \frac{F_j}{2\rho^2} + G_j \tag{35}$$

where F_j and G_j are arbitrary constants.

Making use of (35) one can easily solve the system (7) to get

$$\begin{aligned} w &= \frac{k}{\gamma_j}\left[\frac{F_j}{2(\nu-1)}\ln\rho - \frac{G_j\rho^2}{2(\nu+1)} + \frac{p}{32}\rho^4\right] + H_j, \\ z &= \frac{k}{\gamma_j}\left[\frac{F_j}{2\rho(\nu-1)} - \frac{G_j\rho}{\nu+1} + \frac{p}{8}\rho^3\right], \end{aligned} \tag{36}$$

for $\rho \in S_j$, where $j = 0, 1$. Here F_j, G_j, H_j ($j = 0, 1$) stand for arbitrary constants.

Making use of (35), (36) and the continuity conditions $w(\alpha-) = w(\alpha+)$; $z(\alpha-) = z(\alpha+)$, $m_1(\alpha-) = m_1(\alpha+)$ after algebraic transformations one easily obtains

$$\begin{aligned} G_1 &= \frac{\left[\gamma_0 + \frac{3+\nu}{\alpha^2} - \frac{p}{16}\delta - (\nu-5)\eta^2\right](1-\alpha^2)}{\alpha^2 - \beta(1-\alpha^2)}; \\ F_1 &= 2G - \frac{p}{4}(3+\nu); \\ H_1 &= \frac{kG}{2(1+\nu)\gamma_1} - \frac{kp}{32\gamma_1}; \\ G_0 &= \frac{G}{2\alpha^2}\left[\frac{2\beta\alpha^2\eta^2}{\eta^2-\alpha^2} - 2(1-\alpha^2)\right] - \frac{p}{16\alpha^2}\left\{\frac{\gamma_0}{\gamma_1}[(1+\nu)(3+\nu)- \right. \\ &\left. -\alpha^4(\nu^2-1)] + (\nu-1)[3+\nu-\alpha^4(\nu+1)]\right\}; \\ F_0 &= \frac{2\alpha^2\eta^2 G}{\eta^2-\alpha^2}\beta - \frac{p}{8}\left\{\frac{(1+\nu)\gamma_0}{\gamma_1}[3+\nu+\alpha^4(1-\nu)]+ \right. \\ &\left. +(\nu-1)[3+\nu-\alpha^4(1+\nu)]\right\}; \\ H_0 &= \frac{k}{32(\nu^2-1)}\left\{G\left[\frac{32(1+\nu)}{\gamma_1}\ln\alpha + 16(\nu-1)(1-\alpha^2)\left(\frac{1}{\gamma_0} - \frac{1}{\gamma_1}\right) - \right.\right. \\ &\left. - \frac{16\alpha^2\eta^2}{\gamma_0(\eta^2-\alpha^2)}\beta(2(1+\nu)\ln\alpha - \nu + 1)\right] + \frac{p(\nu-1)}{\gamma_0}[(3+\nu-\alpha^4(\nu+1))\cdot \\ &\cdot(2(\nu+1)\ln\alpha - \nu + 1) - 4 - \alpha^4\gamma_0(\nu+1)] + \frac{p(\nu+1)}{\gamma_1}[(2(\nu+1)\ln\alpha - \\ &\left. -\nu+1)(3+\nu-\alpha^4(\nu+1)) - 4(3+\nu)\ln\alpha + (\nu-1)(\alpha^4-1)]\right\}. \end{aligned} \tag{37}$$

In (37) one has denoted $G = G_1$ and

$$\begin{aligned} \delta &= \frac{\eta^2-\alpha^2}{2\alpha^2\eta^2}\left[(1+\nu)(3+\nu) + \alpha^4(1-\nu)\frac{\gamma_0}{\gamma_1} + (1-\nu)(3+\nu-\alpha^4(1+\nu)\right], \\ \beta &= \frac{\eta^2-\alpha^2}{2\alpha^2\eta^2}\left\{(1-\nu)(1-\alpha^2) + \left[\alpha^2(1-\nu) + 1 + \nu\right]\frac{\gamma_0}{\gamma_1}\right\}. \end{aligned} \tag{38}$$

6 Solution of the adjoint (conjugate) system

In order to solve the adjoint system (23) one can determine from (23)

$$\psi_2 = \frac{\gamma_j}{k}\left(\psi_3' - \frac{1-\nu}{\rho}\right)\psi_3 \tag{39}$$

Substituting ψ_2 and ψ_2' from (39) to (23) leads to the equation

$$\psi_3'' - \frac{\psi_3}{\rho} = -\frac{\psi_1 k}{\gamma_j} \tag{40}$$

which can be integrated to give

$$\psi_3 = -\frac{\psi_1 k}{\gamma_j}\left(\frac{\rho^2}{2}\ln\rho - \frac{\rho^2}{4}\right) + \frac{B_j}{2}\rho^2 + E_j. \tag{41}$$

Accounting for (41) one can solve the rest of the set (23). The solution can be presented as

$$\begin{aligned} \psi_1 &= C, \\ \psi_2 &= \frac{C}{4}\rho[-2\ln\rho(1+\nu)+\nu-1] + \frac{\gamma_j}{k}\left[\frac{\rho}{2}B_j(1+\nu) + \frac{1}{\rho}(\nu-1)E_j\right], \end{aligned} \tag{42}$$

C, B_j, E_j being arbitrary constants of integration. These constants will be determined by the help of boundary conditions $w(1) = 0$, $m_1(1) = 0$ also (9), (10) and continuity requirements at $\rho = \alpha$.

For determination of B_j, C_j, E_j $(j = 0, 1)$ in (41), (42) the tansverslity condition (27) with continuity conditions (28) and requirements (29) - (31) can be used. Note that the relations (29) - (31) provide two algebraic equations with respect to B_j, C_j, E_j after elimination of $\lambda_i = \psi_i(\eta)$ and $w'(\eta)$, $z'(\eta)$, $m_1'(\eta)$ from (30), (31).

7 Discussion

The system of algebraic and differential equations obtained above is solved numerically up to the end. Results of the calculations are presented in Fig. 2 in the case of the plate with a single step of the thickness. In Fig. 2 optimal values of $\alpha = a/R$ and $\gamma = h/h_*$ are presented versus $\frac{w_0}{kp} = \frac{3H(5+\nu)}{32(1+\nu)}$.

8 Concluding remarks

An optimization method relying on the theory of optimal control was developed for circular plates of piece wise constant thickness. The behaviour of the material was assumed to be ideally elastic-plastic. Numerical solution was calculated for simply supported stepped plates with single step of thickness.

The results of calculations showed that considerable economy of material consumption can be achieved when using the design with piece wise constant thickness.

9 References

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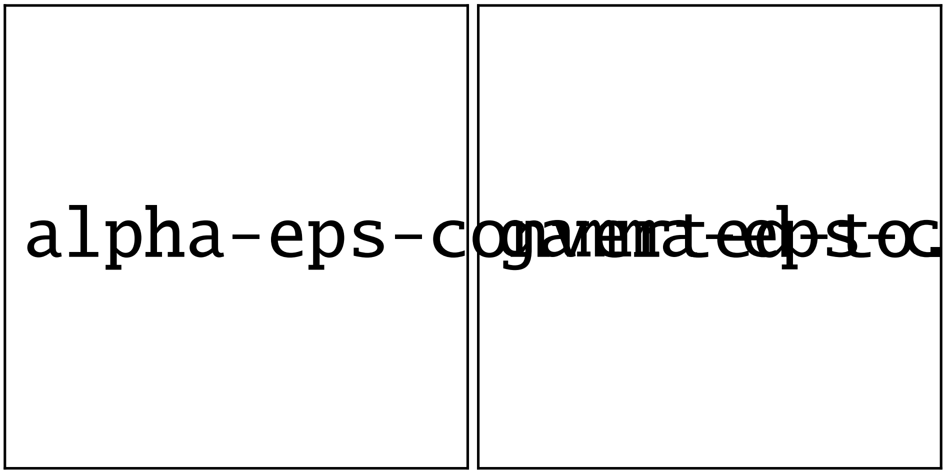

Figure 2: Optimal values of a/R and h/h_*.

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