Elastic plastic bending of stepped annular plates

JAAN LELLEP
University of Tartu
Institute of Mathematics
J. Liivi Street 2, 50409 Tartu
ESTONIA
jaan.lellep@ut.ee

JULIA POLIKARPUS
University of Tartu
Institute of Mathematics
J. Liivi Street 2, 50409 Tartu
ESTONIA
julia.polikarpus@ut.ee

Abstract: Axisymmetric bending of annular plates subjected to the distributed transverse pressure is studied. The plates under consideration have piece wise constant thickness of carrying layers and are fully clamped at the inner edge whereas the outer edge is absolutely free. It is assumed that the material of the plate is an ideal elastic plastic material obeying the square yield condition and the associated flow law in the stage of plastic deformations. Assuming that displacements, generalized stresses and strains remain axisymmetric and making use of the pure bending theory of thin plates the stress strain state of the plate is determined for the initial elastic and subsequent elastic plastic stages of the deformation. Numerical results are presented for a plate with a single step of the thickness.

Key–Words: annular plate, stepped thickness, elasticity, plasticity, associated flow law, yield condition

1 Introduction

The behavior of plates and shells in the range of elastic deformations has been studied by many authors (see [7, 11].

It is well known that the structural material is used in more efficient manner and the ratio of the strength to weight is larger if inelastic deformations are taken into account when designing the structure. The behavior of annular plates made of perfectly plastic materials is studied by Lellep and Mürk (2008) in the case of impulsive loading. Although the early results of the behavior of elastic plastic circular and annular plates have obtained a long ago by Hodge (1981); Tekinalp (1957) most of the attention is paid to plates of constant thickness only. Comprehensive reviews of these investigations can be found in the books by Chakrabarty (2000); Kaliszky (1989); Save, Massonnet, Saxce (1997); Yu, Zhang (1996). Hodge introduced an essential simplification of inelastic problems in the case of a Tresca material making use of the yield surface consisting of two hexagons on the planes of moments and membrane forces, respectively. It was used by many investigators for getting approximate solutions. Among others, Sherbourne, Srivastava (1971) found an analytical solution to the elastic plastic bending problem in the range of large deflections. Lellep and Polikarpus (2008) studied the bending of elastic plastic circular plates of minimum weight in the case of Tresca's yield condition.

In the present paper the elastic plastic bending of annular plates of piece wise constant thickness is studied. It is assumed that the plate is clamped at the inner edge. The material of the plate is ideal elastic plastic obeying the square yield condition.

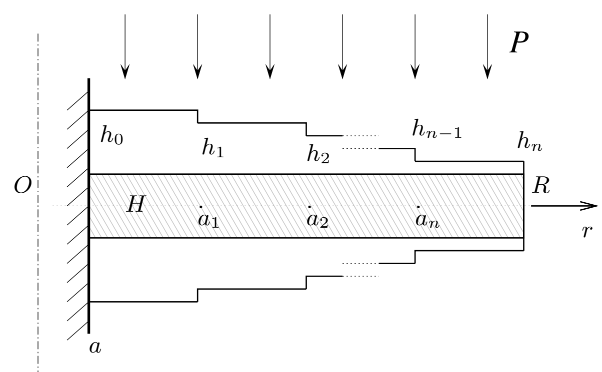

Figure 1: Sandwich cross-section.

2 Problem formulation and basic hypotheses

Let us consider the axisymmetric bending of an annular plate subjected to the transverse pressure of intensity $P = P(r)$, where r is the current radius. Assume that the internal edge of the plate of radius a is clamped whereas the external edge of radius R is absolutely free.

The plate under consideration has sandwich-type

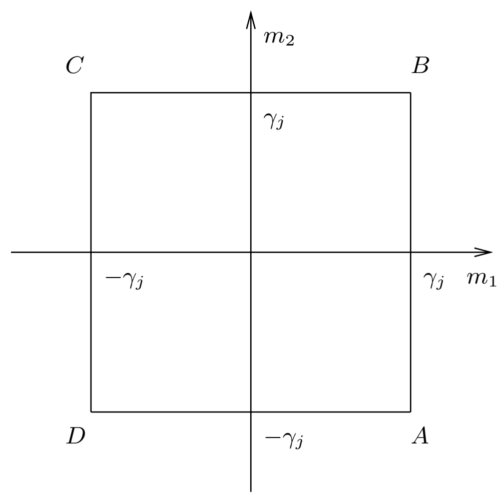

Figure 2: Square yield condition.

cross section. It consists of two rims or carrying layers of thickness h whereas the space between the rims is stuffed with the core material. The latter is not able to resist to normal loads. Let the layer of the core material be of constant thickness H and the carrying layers be of piece wise constant thickness, e. g. (Fig. 1)

$$h = h_j \tag{1}$$

for $r \in (a_j, a_{j+1})$, where $j = 0, 1, \ldots, n$. It is reasonable to denote $a_0 = a$ and $a_{n+1} = R$. The thicknesses $h_0, \ldots, h_n$ and the step locations $a_1, \ldots, a_n$ are assumed to be given geometrical parameters of the plate.

The aim of the paper is to determine the transverse deflection as well as bending moments distributions in the elastic and subsequent inelastic stages of deformation for given transverse pressure levels.

3 Basic equations and concepts

The equilibrium conditions of an element of an axisymmetric plate furnish the equations (see [7, 11])

$$\frac{d}{dr}(rM_1) - M_2 - rQ = 0, \quad \frac{d}{dr}(rQ) = -Pr, \tag{2}$$

provided no external shear loading is applied to the plate. The assumptions of the classical thin plate theory require transverse shear deformations to be zero. However, the shear force Q is taken into account.

The strain components associated with the bending moments M_1, M_2 in the pure bending theory are

$$\kappa_1 = -\frac{d^2W}{dr^2}, \quad \kappa_2 = -\frac{1}{r}\frac{dW}{dr}. \tag{3}$$

It is well known that in the case of lower values of the pressure loading the plate is pure elastic. The elastic behavior of the material can be prescribed with Hooke's law. The latter is to be presented in the generalized form as [7]

$$M_1 = D_j(\kappa_1 + \nu\kappa_2), \quad M_2 = D_j(\kappa_2 + \nu\kappa_1) \tag{4}$$

where $j = 0, \ldots, n$ and in the case of a sandwich plate

$$D_j = \frac{Eh_jH^2}{2(1-\nu^2)}. \tag{5}$$

In (4), (5) and henceforth E and ν denote the Young and Poisson modulus, respectively.

During the subsequent quasistatic increase of the external loading constitutive equations (4) hold good until the elastic limit is exhausted at an unknown point of the plate. In the case of the pressure of constant intensity the yield limit is reached at first at the clamped edge of the plate. After that the plate is subdivided into elastic and plastic regions, respectively. Let these regions be S_e and S_p, respectively. Since we are studying the plate of sandwich type and the carrying layers are thin, no elastic plastic state of a cross section occurs.

It is assumed that the material of the plate obeys the square yield condition and associated flow rule (Fig. 2). Thus, for $r \in S_p$ the stress state is such that the points $(M_1(r), M_2(r))$ lie on a side of the square (Fig. 2). It means that at each point $r \in (a_j, a_{j+1})$ of the plate inequalities

$$|M_1| \leq M_{0j}, \quad |M_2| \leq M_{0j} \tag{6}$$

are satisfied where M_{0j} stands for the yield moment corresponding to the thickness h_j. It can be easily stated that [1]

$$M_{0j} = \sigma_0 h_j H, \tag{7}$$

σ_0 being the yield stress of the material. In an elastic region for $r \in S_e$ inequalities (6) are satisfied as strict inequalities.

Evidently, at the boundary of the plate requirements

$$M_1(R) = 0, \quad Q(R) = 0 \tag{8}$$

and

$$W(a) = 0 \tag{9}$$

must be satisfied at each loading level.

Let us consider the governing equations separately in elastic and plastic regions, respectively. In elastic regions the stress strain state is determined according to (2) and (4). Substituting (4) and (3) in (2) easily leads to the equation

$$\frac{1}{r}\frac{d}{dr}\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dW}{dr}\right)\right]\} = \frac{P(r)}{D_j} \tag{10}$$

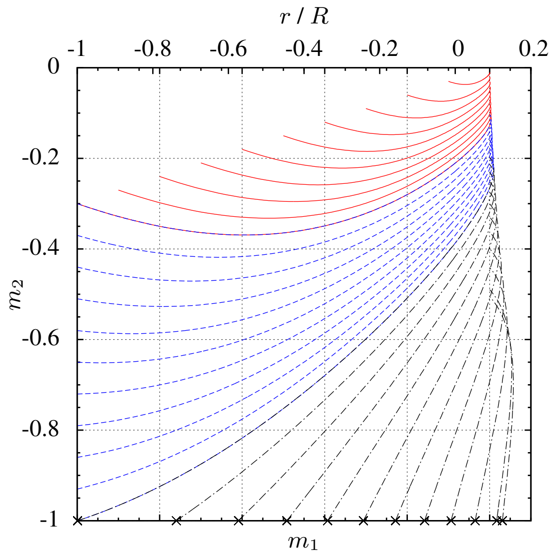

Figure 3: Bending moments m_1, m_2.

for $r \in (a_j, a_{j+1})$, provided $(a_j, a_{j+1}) \subset S_e$.

In the following it is reasonable to use non-dimensional quantities

$$\begin{aligned} &\rho = \frac{r}{R}, \quad m_1 = \frac{M_1}{M_*}, \quad m_2 = \frac{M_2}{M_*}, \\ &q = \frac{RQ}{M_*}, \quad \alpha = \frac{a}{R}, \quad \alpha_j = \frac{a_j}{R}, \\ &p = \frac{PR^2}{M_*}, \quad w = \frac{W}{H}, \quad \gamma_j = \frac{h_j}{h_*}, \\ &d_j = \frac{EH^2 h_j}{2(1-\nu^2)\sigma_0 R^2 h_*} \end{aligned} \tag{11}$$

where $M_* = \sigma_0 h_* H$ is the yield moment of a reference plate of constant thickness h_*.

Making use of variables (11) one can present the equilibrium equations (2) as

$$((\rho m_1)' - m_2)' + p\rho = 0 \tag{12}$$

where prims denote the differentiation with respect to the non-dimensional radius ρ.

4 General solutions in elastic and plastic regions

Let us denote an elastic region (a_j, a_{j+1}) where the thickness of carrying layers is h_j by S_{ej}.

Making use of (10) and (11) it is easy to recheck that the general solution of (10) can be presented as

$$w = A_{1j}\rho^2 \ln\rho + A_{2j}\rho^2 + A_{3j}\ln\rho + A_{4j} + \frac{p\rho^4}{64 d_j} \tag{13}$$

where $\rho \in S_{ej}$ and $d_j = \frac{D_j H}{M_* R^2}$. Arbitrary constants $A_{1j}, \ldots A_{4j}$ will be determined from the boundary requirements and continuity conditions for w, w', m_1 at the boundaries between elastic and plastic regions.

Non-dimensional bending moments can be determined according to (4) and (11) as

$$\begin{aligned} m_1 &= -d_j\left(w'' + \frac{\nu}{\rho}w'\right), \\ m_2 &= -d_j\left(\frac{w'}{\rho} + \nu w''\right) \end{aligned} \tag{14}$$

for $\rho \in S_{ej}$. The terms with derivatives w', w'' in (14) can be expressed as

$$w' = A_{1j}(2\rho\ln\rho + \rho) + 2A_{2j}\rho + \frac{A_{3j}}{\rho} + \frac{p\rho^3}{16 d_j} \tag{15}$$

and

$$w'' = A_{1j}(2\ln\rho + 3) + 2A_{2j} - \frac{A_{3j}}{\rho^2} + \frac{3p\rho^2}{16 d_j} \tag{16}$$

for $\rho \in S_{ej}$.

The third stress component besides bending moments is the shear force. It is reasonable to calculate it from the equilibrium equations (2) or (12). From (12), (2) one easily obtains the equation

$$(\rho q)' = -p\rho \tag{17}$$

which holds good over the entire plate. The solution of this equation which satisfies the boundary condition $q(1) = 0$ is

$$q = -\frac{p}{2}\left(\rho - \frac{1}{\rho}\right) \tag{18}$$

for $\rho \in (\alpha, 1)$. The particular solution of basic equations in a plastic region S_p depends on the particular yield regime. It appears that in the present case the stress strain state in a plastic region of the plate corresponds to the sides AD or DC of the square yield condition (Fig. 2). Let us consider these yield regimes in a greater detail.

In the case of the yield profile CD one has $m_1 = -\gamma_j$ for $\rho \in S_{pj} \subset (a_j, a_{j+1})$. However, it can be rechecked (see [1, 3, 8]) that this regime can not take place at a region of finite length.

If the stress profile lies on the side AD of the yield square (Fig. 2) then $m_2 = -\gamma_j$ and after integration of (12) with (18) one has

$$m_1 = -\gamma_j - \frac{p}{2}\left(\frac{\rho^2}{3} - 1\right) + \frac{E_j}{\rho} \tag{19}$$

for $\rho \in S_{pj}$, where E_j is an arbitrary constant.

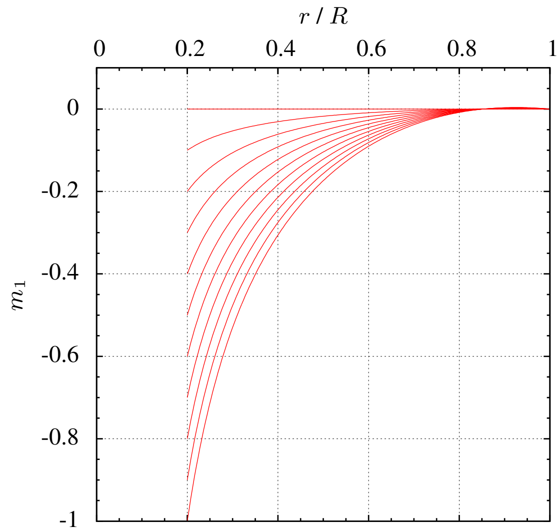

Figure 4: Bending moment m_1; stage I.

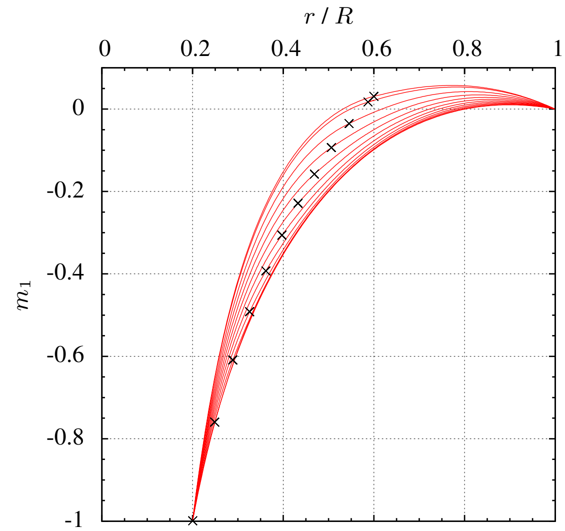

Figure 5: Bending moment m_1; stage III.

According to the flow law in the case of the yield regime AD one has $\kappa_1 = 0$ and thus

$$w = A_j \rho + B_j \tag{20}$$

for $\rho \in S_{pj}$, where A_j, B_j are arbitrary constants. However, on the side CD of the yield condition (Fig. 2) $\kappa_2 = 0$ and thus $w = const$. This is one of the reasons why the regime CD does not take place in a region of finite length.

5 The pure elastic stage of deformation (stage I)

As the intensity of the pressure loading is increased from zero, the entire plate is elastic until the stress profile reaches a side of the yield condition. However, during the elastic stage the stress profile lies inside the square $ABCD$ (Fig. 2) for each $\rho \in (a_j, a_{j+1})$.

During this stage of loading the deflection is defined by (13), bending moments and the shear force by (14) and (18), respectively. For determination of unknown constants one can use the boundary conditions $w'(a) = w(a) = 0$, $m_1(1) = 0$ and the continuity requirements

$$[w(\alpha_j)] = 0, \quad [w'(\alpha_j)] = 0, \quad [m_1(\alpha_j)] = 0 \tag{21}$$

for $j = 1, \ldots, n$ where the square brackets denote finite jumps, e. g. $[z(\alpha_j)] = z(\alpha_j + 0) - z(\alpha_j - 0)$.

It appears that the general solution in the form (13) may involve inadequate solutions for particular cases. In order to avoid this one has to check if the shear force in the form (18) concides with that following from the first equation of the system (2).

Let us denote

$$\rho \bar{q} = (\rho m_1)' - m_2. \tag{22}$$

Evidently, $\bar{q} = q$. Thus one has to check if the constraints

$$\bar{q}(\rho) = q(\rho) \tag{23}$$

for $\rho \in S_{ej}$ ($j = 1, \ldots, n$) with the boundary condition $\bar{q}(1) = 0$ are satisfied. Making use of (13) - (16) and (18) with (22) it is easy to show that equalities (23) take place if

$$A_{1j} = -\frac{p}{8d_j} \tag{24}$$

for $j = 0, \ldots, n$.

Thus, for determination of $3n + 3$ unknown constants A_{2j}, A_{3j}, A_{4j} ($j = 0, \ldots, n$) one has three boundary conditions and $3n$ continuity conditions (21).

The elastic loading stage completes at the moment when the stress profile reaches the side CD of the yield square. In the case of a plate of constant thickness the plastic yielding happens at first at the internal edge of the plate for $\rho = \alpha$. At the boundary between the fully elastic stage and inelastic stage for $p = p_0$

$$m_1(\alpha) = -\gamma_0. \tag{25}$$

Note that, in principle, the plastic yielding may start elsewhere, as well. If, for instance, the inner annulus is very narrow and the thickness h_0 is large whereas the next annulus has very small thickness then the yield can start from the next annulus. However, these cases will not studied in the present paper.

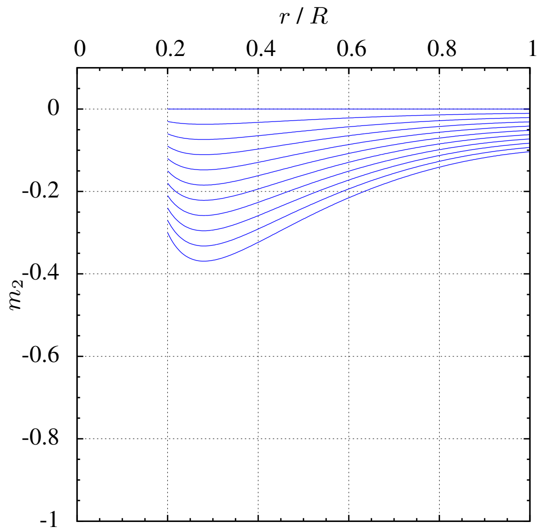

Figure 6: Bending moment m_2; stage I.

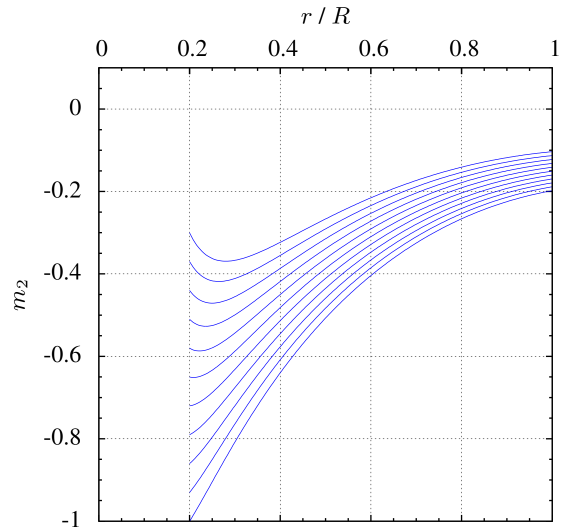

Figure 7: Bending moment m_2; stage II.

6 Elastic plastic stage with the hinge circle (stage II)

Assume that during this stage of deformation the plastic hinge circle is located at the internal edge of the plate and the rest of the plate for $\rho \neq \alpha$ is elastic as during the previous stage. However, due to the hinge the boundary condition $w'(\alpha) = 0$ is no more valid. For determination of unknown constants A_{2j}, A_{3j}, A_{4j} one can use relations (21) - (23) with boundary conditions $w(\alpha) = 0$, $m_1(1) = 0$ and (25). Note that (24) remains valid, as well. The latter admits to present the deflection for $\rho \in [\alpha_j, \alpha_{j+1}]$ as

$$w = \frac{p\rho^2(\rho^2 - 8\ln\rho)}{64d_j} + A_{2j}\rho^2 + A_{3j}\ln\rho + A_{4j}. \quad (26)$$

This stage of determination will be completed when the stress profile at $\rho = \alpha$ reaches the point D (Fig. 2) so that $m_2(\alpha) = -\gamma_0$. Let the corresponding value of the external load intensity be p_1.

7 The elastic plastic stage with a plastic region of finite length (stage III)

It is reasonable to assume that during the subsequent increasing of the transverse pressure plastic deformations take place for $\rho \in S_{p0}$. Assume that $S_{p0} = [\alpha, \eta]$ where η is a previously unknown constant. The plastic region corresponds to the yield regime DA (Fig. 2). Thus, for $\rho \in (\alpha, \eta)$

$$m_2 = -\gamma_0. \quad (27)$$

The distribution of the radial bending moment m_1 can be calculated by (19) taking $j = 0$. As at $\rho = \alpha$, $m_1 = -\gamma_0$ the arbitrary constant E_0 is to be

$$E_0 = \frac{\alpha p}{2}\left(\frac{\alpha^2}{3} - 1\right). \quad (28)$$

Thus the bending moment is defined as

$$m_1 = -\gamma_0 - \frac{p}{2}\left(\frac{\rho^2}{3} - 1\right) + \frac{\alpha p}{2\rho}\left(\frac{\alpha^2}{3} - 1\right) \quad (29)$$

for $\rho \in [\alpha, \eta]$.

8 Numerical results

Results of calculations in the case of plates with a single step are presented in Fig. 3 – 8. The results regard to the plate with inner radius $a = 0.2R$. The stress profiles on the plane of moments m_1, m_2 are shown in Fig. 3 for different values on the load intensity. It can be seen from Fig. 3 that the profiles corresponding to smaller values of the load p lie wholly inside the square $|m_1| \leq 1$, $|m_2| \leq 1$. When the load intensity increases until $p = p_1$ the end of the profile reaches the side $m_1 = -1$ and for $p = p_2$ the corner point where $m_1 = m_2 = -1$. During the subsequent growth of the load intensity the end of the stress profile lies on the side $m_2 = -1$ as it was expected theoretically.

Distributions of the bending moments m_1 and m_2 are presented in Fig. 4 – 5 and Fig. 6 – 8, respectively. The locations of boundaries between elastic and plastic regions in Fig 5 are shown by asterisks. It can be seen from Fig. 7 that when the load increases the stress state tends to the pure plastic state. In the case of a plate of constant thickness in the pure plastic state $m_2 \equiv -1$. In the case of a stepped plate it can be such that $m_2 = -\gamma_j$ for $\rho \in (\alpha_j, \alpha_{j+1})$, $j = 0, \ldots, n$.. However, the question which is the stress state at the plastic collapse can be answered by the limit analysis of the plate of particular shape.

9 Concluding remarks

A method for theoretical investigation of axisymmetric plates subjected to the distributed transverse pressure was developed. The material of plates was assumed to be an ideal elastic plastic material obeying the square yield condition and the associated flow law in the range of inelastic deformations. In order to get maximum simplicity of the posed problem hardening of the material as well as geometrical non-linearity of the plate behavior were neglected.

It was assumed that the plates under consideration had piece wise constant thickness with arbitrary number of steps. Exact solutions were developed for the case when the plate is clamped at the inner edge whereas the outer edge is absolutely free. As a result of the solution procedure a succession of stress states which are in equilibrium with the external loading were constructed that led from the wholly elastic to the elastic plastic state and finally to the plastic collapse state. Since a plate of variable thickness can be approximated with an appropriate choice of the piece wise constant thickness the present solution technique is applicable for approximate solution of similar problems in the cases of plates of variable thickness.

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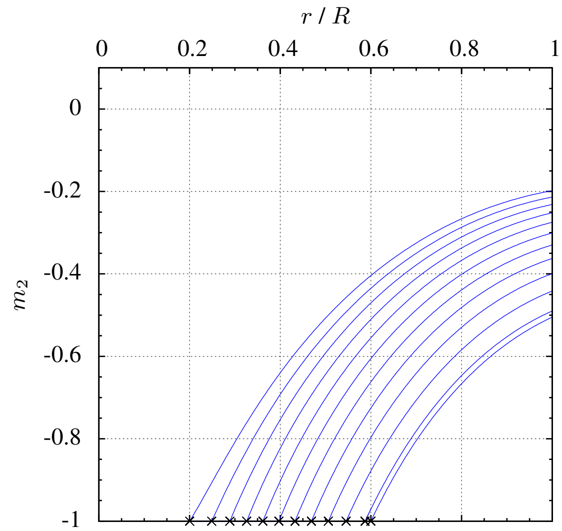

Figure 8: Bending moment m_2; stage III.

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