

JULIA POLIKARPUS

Elastic plastic analysis and optimization of
axisymmetric plates

Faculty of Mathematics and Computer Science, University of Tartu, Tartu, Estonia.

Supervisor: Prof. Jaan Lellep, D. Sc
University of Tartu,
Tartu, Estonia

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INTRODUCTION

The reduction of the structural compliance of thin walled beams, plates and shells is often the primary concern in the engineering mechanics. The need for reduction of the compliance and the increase of the structural stiffness is related to the use of light-weight structures which are less material consuming than the traditional structures.

It is wellknown that the structural material is used in more efficient manner and the ratio of the strength to weight is larger if inelastic deformations are taken into account when designing the structure. Although the early results of the behaviour of elastic plastic circular and annular plates have obtained a long ago by Naghdi [72], Hodge [27], Tekinalp [93] the most of the attention is paid to plates of constant thickness only. Comprehensive reviews of these investigations can be found in the books by Chakrabarty [11], Cohn [15], Kaliszky [35], Save et al [86], Yu and Zhang [109]. Hodge [29] introduced an essential simplification of inelastic problems in the case of a Tresca material making use of the yield surface consisting of two hexagons on the planes of moments and membrane forces, respectively. It was used by many investigators for getting approximate solutions. Among others, Sherbourne, Srivastava [89] found an analytical solution to the elastic plastic bending problem in the range of large deflections.

The exact analysis of elastic plastic plates is quite complicated. This involves the need for physically reasonable simplifications. An effective simplification is introduced by Haythornthwaite [25] and Tekinalp [93, 94]. Haythornthwaite suggested to assume that any plate element is either entirely elastic or entirely plastic. This assumption is fulfilled in the case of a sandwich plate consisting of two carrying layers and of a core material between the rims. Haythornthwaite [25] investigated the elastic plastic bending of an annular plate loaded by the central absolutely rigid boss. The plate is clamped to the boss at the inner edge and simply supported at the outer edge. An exact theoretical solution is derived for a plate made of a Tresca material.

Tekinalp [93] studied the circular plate simply supported at the edge and subjected to the uniformly distributed transverse pressure. The solution for a plate clamped at the outer edge was obtained a little later by Tekinalp [94]. The solutions in both cases consist of the initial elastic stage and several subsequent elastic plastic stages of deformation.

The complete elastic plastic analysis of centrally clamped annular plates carrying uniformly distributed transverse loading was undertaken by French [19] within the frame works of the pure bending theory of thin plates. Exact stress profiles lying partly inside and partly on the yield hexagon are developed for the elastic and elastic plastic stages of the deformation. It is assumed by French [19] that the plate is a

sandwich construction and that the material of carrying layers obeys the Tresca yield condition. The elastic plastic bending of circular and annular plates made of a Tresca material was also studied by Hodge [27, 28].

Elastic plastic deformations of axisymmetric plates made of a von Mises material are investigated by Eason [18], Sokolovsky [90], Lackman [39], Popov et al [80], Turvey [96], Turvey and Lim [97], Turvey and Salehi [98], Ohashi, Murakami and Endo [74, 75].

The classical concept of an elastic plastic body admits the exact definition of yielding and of the yield-point load. It is shown earlier by Tekinalp [93], Hodge [27] and others that in the case of the Tresca material and a plate subjected to the uniformly distributed transverse pressure of intensity p the onset of yielding corresponds to the values of the pressure $p = \frac{16M_0}{R^2(3 + \nu)}$ and $p = \frac{8M_0}{R^2}$ for simply supported and clamped plates, respectively. Here ν is the Poisson ratio, R and M_0 stand for the radius of the plate and the yield moment of the material, respectively. The concept of gradual yielding introduced by Richard and Abbott [84] for one-dimensional structures was extended to circular and annular plates by Upadrasta et al [100] and Khalili and Peddieson [36].

Elastic plastic bending of axisymmetric plates in the range of large deflections was investigated by Ohashi et al [75], Sherbourne and Srivastava [89], also Gorji and Akileh [23], Turvey [95]. Stress and strain distributions in rotating disks made of a Tresca material are determined by Güven [24] and Gamer [20]. In the cited papers solid and sandwich plates and rotating disks are investigated assuming that the material is an ideal elastic plastic material without strain hardening. Plates made of a hardening material were studied by Boyce [8], Hwang [31], Tanaka [92]. Wen [108], Kirs [37] considered work-hardening circular plates subjected to dynamic loading under different circumstances.

The use of the classical bending theory based on Kirchhoff hypotheses is justified in the case of thin plates and homogeneous materials. In more complicated cases the shear stresses must be taken into account as shown by Oblak [73], Nagai and Ito [71]. Sawczuk and Duszek [87] developed similar concept for ideal plastic plates.

Wang et al [104, 105] established designs of rigid-plastic solid plates with a maximum thickness constraint.

Reddy and Wang [83] investigated relationships between the classical pure bending theory and the shear deformation theory applied to the bending of circular and annular plate. The full description of the shear deformation theory of plates and shells can be found in the book by Wang et al [103]. A variational approach to shear deformable plates within gradient elasticity is developed by Mousavi and Paavola [68, 67].

One of the ways of increasing the stiffness of beams, plates and shells is to furnish

these structural elements with additional supports. Evidently, it is reasonable to settle these supports at the optimal positions.

The problem of minimization of the compliance of elastic beams and the determination of the optimal location of the additional support was first formulated by Mróz and Rozvany [70]. In paper Mróz and Rozvany [70] designs of minimum compliance of beams are established in the case of quasistatic loading. Later Szelag and Mróz [91], Akesson and Olhoff [1], Chou et al [14] treated the problems of maximal eigenfrequency for given stiffness with respect to the location of the additional support. Bojczuk and Mróz [6] developed a new method for simultaneous optimization of topology, configuration and cross-sectional dimensions of elastic beams and beam structures extending earlier results by Garstecki and Mróz [21], Mróz and Lekszycki [69], also by Lepik [62]. In the subsequent papers by Bojczuk and Mróz [7] this concept was applied for optimal design of active supports with force actuators. Olhoff and Akesson [77] treated the stability of columns and Wang et al [106] studied the buckling of axisymmetric plates.

A lot of attention has been paid in the literature to the optimization of internal supports to beam, plate and shell structures in the case of inelastic materials. Probably the first paper in this area is due to Prager and Rozvany [81]. Systematic reviews of results obtained in earlier papers are presented by Rozvany [85], also by Lellep and Lepik [47]. Optimal designs of circular cylindrical shells with additional supports are established by Lellep [42, 45] in the case of dynamic loading and an ideal plastic material. The behaviour of geometrically non-linear cylindrical shells with internal supports is studied in [44, 43, 45, 48].

Optimal designs of axisymmetric plates and shells of various shape made of elastic and inelastic materials are established in [51, 56, 57, 58, 59]. Inelastic spherical and conical shells are studied in [56, 57, 58] whereas a stress strain analysis of an annular plate made of an elastic plastic material is presented in [50].

A design sensitivity analysis for the deflection of beam or plate structures was undertaken by Wang [107] in the case of simple supports located at given mesh nodes. Stiffened sector plates are studied in [98].

In the present work an analytical method of determination of positions of rigid ring supports for circular plates is developed. The analysis is confined to the axisymmetric response of elastic plates to subjected loads.

Plates and shells are widely used in various fields of technology and engineering. That is why it is very important to strive for the optimal shape of the plates, also for the optimal lay out of supports and for the optimal distributions of stresses in the structures. This enables to save the material of structures, provided the optimized structure is able to carry the same loads as the original sample. The behaviour of plates and shells in the range of elastic deformations has been studied by many authors (see

Reddy [82], Vinson [102], Ventsel [101]).

The practical need for light weight structures has increased the importance of investigations of composite materials, also sandwich plates and shells and of these manufactured from composites. The foundations of the analysis and design of sandwich structures was developed by Allen [2] and Plantema [79]. The stability problem of a three-layered plate with a soft core was solved by Pawlus [78]. The critical loads are obtained analytically and numerically. Elastic circular sandwich plates subjected to the ring load have studied by Magnucki et al [65]. The comparison of the analytical, numerical and experimental results reveals small discrepancies between theoretical predictions and experimental data. These are probably due to the approximate determination of the mechanical properties of the core material. The influence of shear forces on the bending of circular plates was studied by Reddy and Wang [83].

There exists a large number of books dedicated to the foundations of composite materials. It is worthwhile to mention among others the books by Herakovich [26], Jones [33], Daniel and Ishai [16], Gibson [22], Andreev and Nemirovskii [3], Tuttle [99].

Researchers and engineers encounter often the need for the optimization of structural elements in order to produce cheaper details and save materials. The problems of optimization of thin-walled plates and shells are investigated by Banichuk [4], Bendsøe [5], Kirsch [38], Cherkaev [13], Melerski [66], Olhoff [76, 12] in the case of elastic materials. The essential benchmark problems of this kind are presented in the books by Lellep [46], Brandt [9].

The optimization of structural elements made of inelastic materials got its start by establishing the Drucker-Shield criterion in the form $\frac{D}{h} = const$ where D is the dissipation of the internal energy and h stands for the thickness of the plate (Drucker and Shield [17]). The Drucker-Shield criterion is based on the lower and upper bound theorems of limit analysis. In its original form it holds good in the case of ideal rigid plastic structures. This approach to the optimal design of structural elements is called "the optimality criteria-based design".

Later various other approaches based on finite element method (FEM), variational methods, principle of maximum and other methods are developed.

Lepik [60, 61] is a pioneer in the application of the principle of maximum in the optimal design of rigid-plastic beams, plates and shells. In [60, 61] optimal designs of inelastic circular plates are established for the collapse load assuming that the thickness of the plate is constrained by given upper and lower bounds. Lepik and Mróz [63, 64] developed new approaches to the optimal design of beams, plates and shells subjected to the dynamic loading. The both cases, the impulsive loading and the rectangular pressure loading of beams with piecewise constant thickness are

investigated and the optimal designs corresponding to different optimality criteria are developed.

Resorting to the principle of maximum optimal designs of circular plates subjected to the concentrated load are developed by Lellep [40].

An approach to the optimal design of geometrically non-linear inelastic beams and cylindrical shells is developed by Lellep [41, 44, 45] making use of the variational methods of the theory of optimal control. This method appeared to be very effective in the determination of optimal locations of additional supports to beams and cylindrical shells [42, 43].

The current thesis is devoted to the analysis and optimization of circular and annular plates made of elastic plastic materials. It is assumed that the material obeys a piecewise linear yield condition and the associated gradientality law. The dissertation consists of five chapters.

In the first chapter of the thesis the governing equations for the analysis of axisymmetric plates are presented, provided the material is an ideal elastic plastic material.

In the second chapter the bending analysis of elastic plastic circular and annular plates is undertaken in the cases of the Tresca's yield condition and the diamond condition, respectively. The results of Chapter 2 are published in [49, 53, 55].

The elastic plastic response of an annular plate to the transverse pressure is studied in the third chapter. It is assumed that the plate is clamped at the inner edge and absolutely free at the outer edge. The results of Chapter 3 are published in [50].

In the fourth chapter an optimization procedure is developed for circular plates with additional ring supports. The results of Chapter 4 are published in [51, 52].

The fifth chapter of the thesis is devoted to the analysis and optimization of stepped plates made of an anisotropic composite material. The results of Chapter 5 are published in [54].

CHAPTER 1

BASIC EQUATIONS AND CONCEPTS

In this chapter the governing equations for thin plates are presented. This system of equations consists of the equations of equilibrium and of the physical relations for elastic and plastic regions for the thin-walled circular and annular plates. For the sake of simplicity a thin-walled plate with constant thickness is considered.

1.1 Equations of equilibrium

We consider axisymmetric deformations of a circular plate subjected to the axisymmetric transverse loading of intensity $P = P(r)$. Here r is the current radius e. g. the distance from the center of the plate. As we are studying the axisymmetric response of the plate all points lying at the circle with radius r have common displacements $W(r)$ in the transverse direction as well as common deformations and curvatures κ_1 , κ_2 in the radial and circumferential directions, respectively. Note that the radial displacement will be neglected in the present study since the classical equations of the bending theory of thin plates will be used.

In the present study we shall employ the linear theory of thin plates (see Reddy

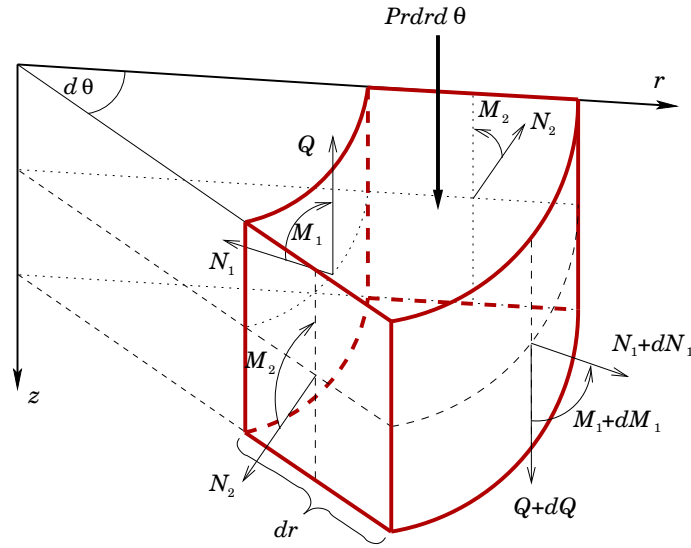


Figure 1.1: An element of the circular plate.

[82], Vinson [102]). According to this approach one can treat the equilibrium of internal and external forces and couples on the basis of an undeformed element of the plate. Let M_1, M_2 be the generalized couples called bending moments in the radial and circumferential directions, respectively. Bending moments M_1, M_2 are the only generalized stress components contributing to the internal energy. Note that the membrane forces are assumed to be small so that one can neglect the membrane action of internal forces. Although the shear force Q may be finite it does not contribute to the internal energy in the classical plate theory. The reason is that the corresponding strain component vanishes.

In the frameworks of the classical plate theory where axial symmetry is retained the couples M_1, M_2 with forces Q and P form a system of forces and moments which keep the element of the plate in equilibrium. The equilibrium conditions of the plate element presented in Fig. 1.1 can be written as

$$\begin{aligned}\frac{d}{dr}(rN_1) - N_2 &= 0, \\ \frac{d}{dr}(rM_1) - M_2 - rQ &= 0, \\ \frac{d}{dr}(rQ) + P(r)r &= 0.\end{aligned}\tag{1.1}$$

Note that in the following the influence of membrane forces N_1, N_2 will be neglected.

1.2 Hooke's law

If the material of the plate is an ideal elastic material the Hooke's law holds good. In the case of an isotropic elastic material The Hooke's law reads

$$\begin{aligned}\varepsilon_1 &= \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)]; & \gamma_{12} &= \frac{\tau_{12}}{G}; \\ \varepsilon_2 &= \frac{1}{E}[\sigma_2 - \nu(\sigma_1 + \sigma_3)]; & \gamma_{13} &= \frac{\tau_{13}}{G}; \\ \varepsilon_3 &= \frac{1}{E}[\sigma_3 - \nu(\sigma_1 + \sigma_2)]; & \gamma_{23} &= \frac{\tau_{23}}{G}.\end{aligned}\tag{1.2}$$

Here ε_j, γ_j ($j = 1, 2, 3$) stand for strain components whereas σ_j, τ_j ($j = 1, 2, 3$) are stress components; E is the Young modulus, ν is Poisson ratio and $G = \frac{E}{2(1 + \nu)}$.

In the case of plane stress state when $\varepsilon_3 = 0$, $\tau_{23} = \sigma_3 = 0$ one has

$$\begin{aligned}\varepsilon_1 &= \frac{1}{E}(\sigma_1 - \nu\sigma_2), \\ \varepsilon_2 &= \frac{1}{E}(\sigma_2 - \nu\sigma_1), \\ \gamma_{12} &= \frac{\tau_{12}}{G}\end{aligned}\tag{1.3}$$

or in the inverted form as

$$\begin{aligned}\sigma_1 &= \frac{E}{1 - \nu^2}(\varepsilon_1 + \nu\varepsilon_2), \\ \sigma_2 &= \frac{E}{1 - \nu^2}(\varepsilon_2 + \nu\varepsilon_1), \\ \tau_{12} &= G\gamma_{12}.\end{aligned}\tag{1.4}$$

The strain components can be expressed via displacements u, v, w in the direction of coordinate axes, respectively. Making use of polar coordinates (now u is directed in the radial direction and v in the circumferential direction) one has

$$\begin{aligned}\varepsilon_1 &= \frac{\partial u}{\partial r}, \\ \varepsilon_2 &= \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \Theta}, \\ \gamma_{12} &= \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \Theta}.\end{aligned}\tag{1.5}$$

In the case axisymmetric loading it is expected that the stress-strain state is also axisymmetric, provided the boundary conditions at the edges of the plate are axisymmetric. Now the displacements do not depend on the polar angle Θ and one has

$$\begin{aligned}\varepsilon_1 &= \frac{du}{dr}, \\ \varepsilon_2 &= \frac{u}{r}, \\ \gamma_{12} &= \frac{dv}{dr} - \frac{v}{r}.\end{aligned}\tag{1.6}$$

According to Kirchhoff hypotheses one has (see Reddy [82], Vinson [102])

$$\begin{aligned}\varepsilon_1 &= z\kappa_1, \\ \varepsilon_2 &= z\kappa_2\end{aligned}\tag{1.7}$$

where z is the distance between a current point and the middle surface of the plate. Principal curvatures \varkappa_1, \varkappa_2 can be expressed as

$$\begin{aligned}\varkappa_1 &= -\frac{d^2 W}{dr^2}, \\ \varkappa_2 &= -\frac{1}{r} \frac{dW}{dr}\end{aligned}\tag{1.8}$$

where W is the transverse deflection or displacement in the z -axis of points lying on the middle surface.

Let us introduce the generalized stresses (bending moments)

$$\begin{aligned}M_1 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_1 z dz, \\ M_2 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_2 z dz, \\ M_{12} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{12} z dz\end{aligned}\tag{1.9}$$

and the shear force

$$Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{13} dz.\tag{1.10}$$

Substituting equations (1.4) with (1.7), (1.8) in (1.9) after integration one has

$$\begin{aligned}M_1 &= D(\varkappa_1 + \nu \varkappa_2), \\ M_2 &= D(\varkappa_2 + \nu \varkappa_1)\end{aligned}\tag{1.11}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}\tag{1.12}$$

in the case of a solid plate and

$$D = \frac{EhH^2}{2(1-\nu^2)}\tag{1.13}$$

in the case of a sandwich plate. In (1.13) H stands for the total thickness and h is the thickness of carrying layers.

The stress strain state is determined according to equations (1.1). Substituting equations (1.11), (1.8) in formula (1.1) leads to a fourth order equation with respect

to the deflection W known from the theory of elastic plates (see Reddy [82], Vinson [102], Ventsel and Krauthammer [101])

$$\frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dW}{dr} \right) \right] \right\} = \frac{P(r)}{D}. \quad (1.14)$$

The general solution of this equation can be presented in the case $P = \text{const}$ as

$$W = \frac{Pr^4}{64D} + A_1 r^2 \ln r + A_2 r^2 + A_3 \ln r + A_4 \quad (1.15)$$

where $A_1 - A_4$ are the integration constants.

1.3 Yield criteria

The theories of plasticity are based on experimental observations regarding the behaviour of inelastic materials. The experiments show that a rod in a simple tension test remains elastic until the stress does not exceed a limit value, called the yield stress σ_0 . In this thesis it is assumed that the stress strain curve can be approximated by two straight lines shown in Fig. 1.2. Thus, according to this model a one-dimensional body (a rod under tension) remains elastic if $|\sigma| < \sigma_0$.

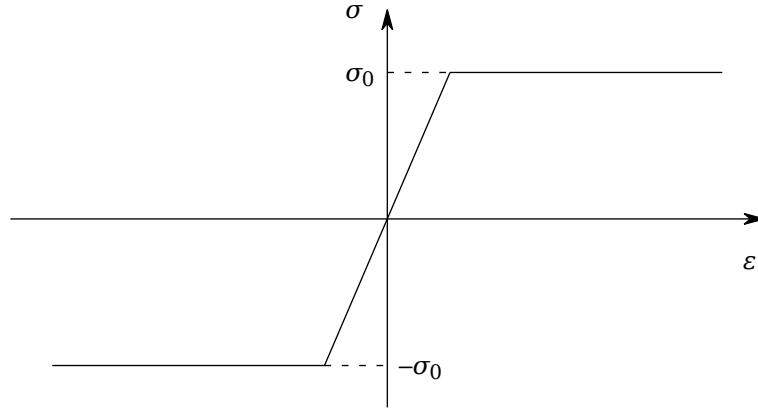


Figure 1.2: The stress-strain curve.

However, in the general case the stress state of a material element can be represented by a point in a nine-dimensional stress space with coordinates σ_{ij} ($i, j =$

1, 2, 3). Theoretical considerations and experimental investigations show that around the origin of the stress space there exists a closed convex surface

$$\Phi_0(\sigma_{11}, \sigma_{12}, \dots, \sigma_{33}) = 0 \quad (1.16)$$

surrounding the region where elastic deformations take place. The points lying on this surface correspond to the points of the body where plastic deformations occur.

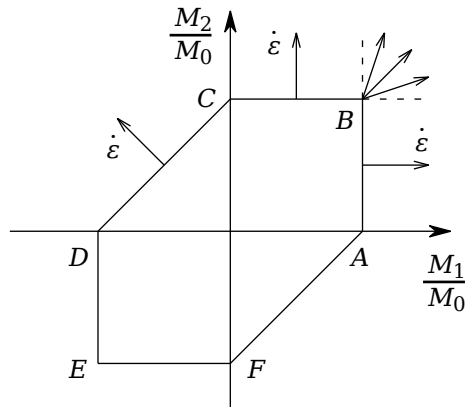


Figure 1.3: Tresca's yield hexagon.

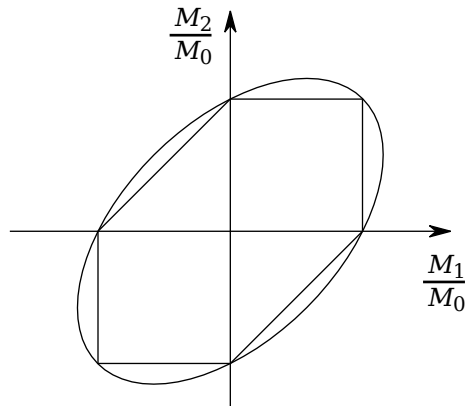


Figure 1.4: Von Mises yield ellipse.

In the case of axisymmetric thin-walled plates the yield surface (1.16) can be expressed via bending moments as

$$\Phi(M_1, M_2, M_0) = 0 \quad (1.17)$$

where M_0 stands for the yield moment. Evidently

$$M_0 = \frac{\sigma_0 h^2}{4} \quad (1.18)$$

for solid plates and

$$M_0 = \sigma_0 h H \quad (1.19)$$

for sandwich plates.

The most often used yield conditions are the Tresca (Fig. 1.3) and Mises (Fig. 1.4) conditions.

1.4 Associated flow rule

In the general case the yield criterion $\Phi \leq 0$ holds good. In an elastic region one has $\Phi < 0$ and in a plastic region $\Phi = 0$. In the theory of plasticity it is shown that in a plastic region of a body where $\Phi = 0$ the strain rate vector $\dot{\epsilon}$ is directed outwards the yield surface (see Chakrabarty [11], Kaliszky [35], Sawczuk and Sokół-Supel [88]). Thus

$$\dot{\epsilon}_{ij} = \frac{\lambda \partial \Phi}{\partial \sigma_{ij}} \quad (1.20)$$

for $i, j = 1, 2, 3$. In equation (1.20) λ stands for a non-negative scalar multiplier and the dot denotes the derivative with respect to time. Using the generalized stresses and the yield surface in the form (1.17) one has

$$\begin{aligned} \dot{\kappa}_1 &= \frac{\lambda \partial \Phi}{\partial M_1}, \\ \dot{\kappa}_2 &= \frac{\lambda \partial \Phi}{\partial M_2} \end{aligned} \quad (1.21)$$

where κ_1, κ_2 stand for the curvatures.

At the non-regular points of the yield surface the strain rate vector is formed as a linear combination with unknown coefficients of outward normals to the adjacent portions of the yield surface.

Note that in the "deformation theory" of plasticity instead of strain rate the strain components themselves are used.

Let us consider the case when the behaviour of the material in plastic stage corresponds to Tresca's yield condition (Fig. 1.3) and to associated gradientality law in a

greater detail. The latter means that if the stress point is lying at an edge of Tresca's hexagon then the vector of curvatures with components (1.8) is directed towards the external normal to the edge. It will be shown that in the case of a circular plate subjected to unidirectional transverse loading the yield regime $M_2 = M_0$ takes place, where M_0 stands for the yield moment.

If the stress state of the plate corresponds to an internal point of the side BC of the yield hexagon, then according to the gradientality law $\dot{\kappa}_1 = 0$, $\dot{\kappa}_2 \geq 0$. Thus

$$\frac{d^2 \dot{W}}{dr^2} = 0 \quad (1.22)$$

in this case. Making use of the deformation theory the general solution of the equation (1.22) can be presented as

$$W = Ar + B \quad (1.23)$$

where A and B are the constants of integration.

If the stress strain state of the plate in certain region corresponds to a non-regular point of the yield curve (for instance, the corner point B in Fig. 1.3), then the strain rate vector lies inside the angle formed by external normals to crossing sides.

At the point B of the Tresca hexagon (Fig. 1.3) $M_1 = M_2 = M_0$ and the associated yield law states that $\dot{\kappa}_1 \geq 0$, $\dot{\kappa}_2 \geq 0$. Thus the exact expressions of strain rates remain unknown here.

Note that if the stress strain state corresponds to an interior point of the yield hexagon (Fig. 1.3) then the plate material remains elastic in this region and Hooke's law holds good.

CHAPTER 2

BENDING OF ELASTIC PLASTIC PLATES

2.1 Preliminaries and basic hypotheses

Let us consider the quasistatic behaviour of an elastic plastic circular plate of radius R under the lateral pressure of intensity P . It is assumed that the plate is of sandwich-type consisting of carrying layers of thickness h and of a core material between the rims. The thickness of the layer of the core material is H . We assume that

$$h = h_j, \quad (2.1)$$

for $r \in (a_j, a_{j+1})$; $j = 0, \dots, n$. The quantities a_j, h_j are treated as preliminary known parameters. For the sake of convenience we take $a_0 = 0$; $a_{n+1} = R$ (see Fig. 2.1). The response of the plate to the external loading will be prescribed by the classical plate theory. The generalized stress components contributing to the strain energy are the bending moments M_1, M_2 in the radial and hoop direction defined by (1.9). Corresponding strain components are the curvatures \varkappa_1, \varkappa_2 which can be determined via the transverse deflection $W = W(r)$. It is assumed that the radial

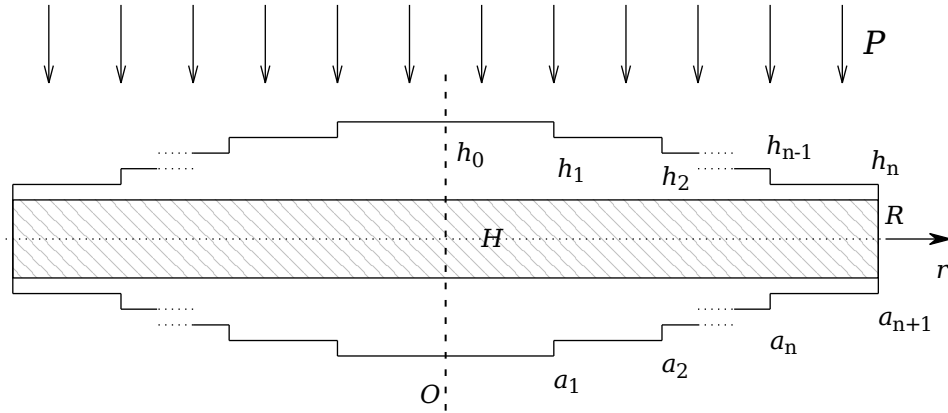


Figure 2.1: Cross-section.

bending moment M_1 as well as the shear force Q , deflection W and slope dW/dr are continuous for each $r \in (0, R)$. However, the deflection slope may have discontinuities at sections where $M_1 = \pm M_0$ (see Chakrabarty [11], Kachanov [34], Yu and

Zhang [109]). It means that

$$\begin{aligned} [W(a_j)] &= 0, \\ [M_1(a_j)] &= 0 \end{aligned} \quad (2.2)$$

and

$$(|M_1(a_j)| - M_0) \left[\frac{dW(a_j)}{dr} \right] = 0. \quad (2.3)$$

In (2.2) and henceforth the square brackets denote finite jumps of discontinuous variables at the given point. According to this notation

$$[z(a)] = z(a + 0) - z(a - 0) \quad (2.4)$$

where $z(a \pm 0)$ stand for right-hand and left-hand limits of the variable z , respectively.

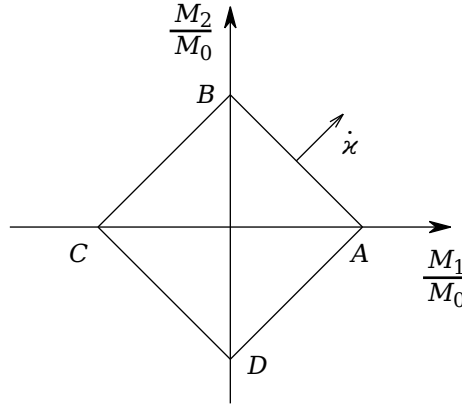


Figure 2.2: Diamond yield condition.

The aim of this chapter is to determine the stress strain state of the plate for the initial elastic and subsequent elastic plastic stages of loading.

If the load intensity is high enough plastic deformations occur in certain regions. Now the plate is divided into elastic and plastic regions. Let us denote these regions S_e and S_p , respectively. In a plastic region the stress state corresponds to a point lying on the yield surface (or a yield curve). It is assumed that the yield condition can be presented by the diamond $ABCD$ shown in Fig. 2.2. Here M_{0j} denotes the yield moment. In the case of a sandwich plate with the rim thickness h_j

$$M_{0j} = \sigma_0 h_j H, \quad (2.5)$$

σ_0 being the yield stress of the material. Since $M_1 \geq 0$, $M_2 \geq 0$ in the most cases one can assume that in the plastic region

$$M_1 + M_2 = M_{0j} \quad (2.6)$$

for $r \in (a_j, a_{j+1})$. Note that the diamond yield condition was suggested by Jones [3] for approximate solution of dynamic problems of plastic plates.

According to the associate flow law on the side AB of the diamond one has $\dot{\varkappa}_1 = \dot{\varkappa}_2$ where dots denote the differentiation with time or a time-like parameter. Making use of (1.8) and the deformation-type theory of plasticity the gradientality law results in the equation

$$\frac{d^2 W}{dr^2} - \frac{1}{r} \frac{dW}{dr} = 0. \quad (2.7)$$

2.2 Integration of governing equations in elastic regions

Assume that the portion of the plate for $r \in (a_j, a_{j+1})$ is in pure elastic stress strain state. For determination of stresses, strains and displacement one has the equations (1.1)–(1.11). Substituting (1.11) with the help of (1.8) in the equilibrium equations (1.1) results in

$$\frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dW}{dr} \right) \right] \right\} = \frac{P(r)}{D_j} \quad (2.8)$$

for $r \in (a_j, a_{j+1})$ where

$$D_j = \frac{E h_j H^2}{2(1 - \nu^2)}. \quad (2.9)$$

This is true under the condition that this interval belongs to the set S_e .

One can easily recheck that the general solution of the equation (2.8) is

$$W = A_{1j} r^2 \ln r + A_{2j} r^2 + A_{3j} \ln r + A_{4j} + \frac{P r^4}{64 D_j} \quad (2.10)$$

where $A_{1j} - A_{4j}$ are arbitrary constants.

Calculating the bending moments according to (1.11) and making use of (1.8), (2.10)

one eventually obtains

$$\begin{aligned}
M_1 &= -D_j \left\{ \frac{Pr^2(3+\nu)}{16D_j} + A_{1j}[2\ln r + 3 + \nu(2\ln r + 1)] \right. \\
&\quad \left. + 2A_{2j}(1+\nu) + \frac{A_{3j}(\nu-1)}{r^2} \right\}, \\
M_2 &= -D_j \left\{ \frac{Pr^2(1+3\nu)}{16D_j} + A_{1j}[2\ln r + 1 + \nu(2\ln r + 3)] \right. \\
&\quad \left. + 2A_{2j}(1+\nu) + \frac{A_{3j}(1-\nu)}{r^2} \right\}.
\end{aligned} \tag{2.11}$$

2.3 Elastic stage of deformation

In the case of smaller values of the intensity of the transverse pressure the plate remains elastic. In the elastic stage the stress strain state of the plate is defined by (2.10) and (2.11). Let us consider now a particular case of the problem posed above when $n = 1$ and the thickness distribution is

$$h = \begin{cases} h_0, & r \in [0, a] \\ h_1, & r \in [a, R]. \end{cases} \tag{2.12}$$

Let the applied loading be of constant intensity. For the concreteness sake let us assume that the plate is simply supported at the edge. Thus at the boundary of the plate

$$\begin{aligned}
M_1(R) &= 0, \\
W(R) &= 0.
\end{aligned} \tag{2.13}$$

According to (2.1) and (2.10) – (2.12) one has to take $j = 0$, if $r \in [0, a]$ and $j = 1$, if $r \in [a, R]$.

Calculating the shear force for the elastic plate one reaches to the relation

$$Q = \frac{1}{r} \left(\frac{d}{dr}(rM_1) - M_2 \right). \tag{2.14}$$

Substituting M_1, M_2 from (2.11) in (2.14) one can see that the shear force is continuous all over the plate if constants $A_{1j} = 0$.

Evidently, $M_1(0)$ must be finite. Thus it follows from (2.10) that $A_{30} = 0$. For determination of the rest unknown constants $A_{20}, A_{40}, A_{21}, A_{31}, A_{41}$, one can use

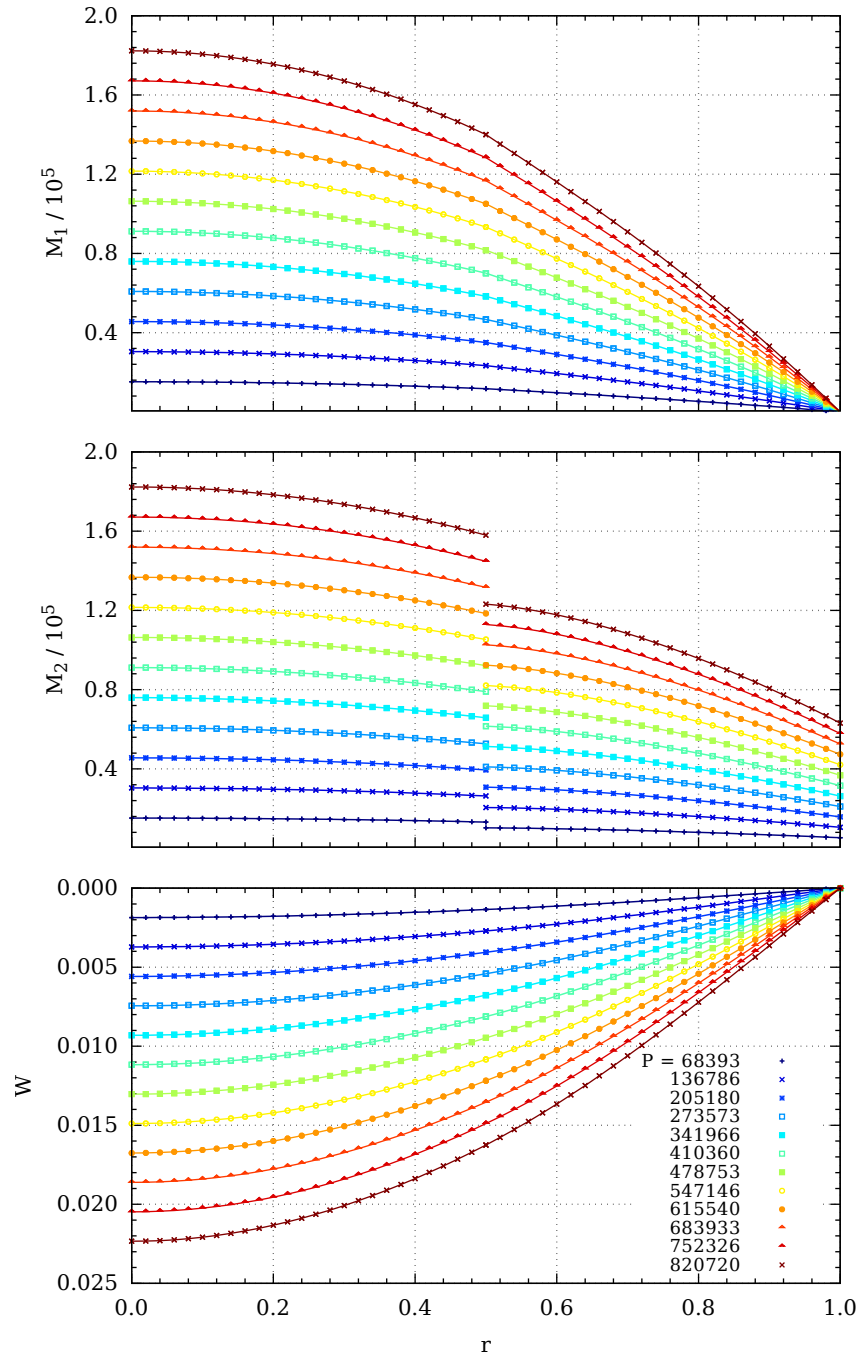


Figure 2.3: Bending moments and deflections of the stepped elastic plate.

the boundary conditions (2.13) and the continuity requirements (2.2), (2.3). Since during the elastic stage $M_1 \neq M_0$ one has

$$\begin{aligned} [W(a)] &= 0, \\ [M_1(a)] &= 0, \\ \left[\frac{dW(a)}{dr} \right] &= 0. \end{aligned} \quad (2.15)$$

From the first equation in (2.13) one can express the constant

$$A_{21} = \frac{-PR^2(3+\nu)}{32D_1(1+\nu)} + \frac{A_{31}(1-\nu)}{2R^2(1+\nu)}. \quad (2.16)$$

Substituting the constant A_{21} from equation (2.16) in the second equation in (2.13) one can determine

$$A_{41} = \frac{PR^4(5+\nu)}{64D_1(1+\nu)} - \frac{A_{31}[1-\nu+2(1+\nu)\ln R]}{2(1+\nu)}. \quad (2.17)$$

Making use of the continuity of the bending moment M_1 at $r = a$ in (2.15) and equation (2.16) one can express

$$A_{20} = \frac{-PR^2(3+\nu)}{32D_0(1+\nu)} + \frac{A_{31}D_1(1-\nu)(a^2-R^2)}{2D_0(1+\nu)R^2a^2}. \quad (2.18)$$

Substituting the constants A_{20} , A_{21} and A_{41} in the first equation in (2.15) one can obtain

$$\begin{aligned} A_{40} &= \frac{P[a^4(1+\nu)(D_0-D_1)+R^4D_0(5+\nu)+2R^2a^2(3+\nu)(D_1-D_0)]}{64D_0D_1(1+\nu)} \\ &+ \frac{A_{31}\{(1-\nu)[(R^2-a^2)D_1+a^2D_0]+D_0R^2[2(1+\nu)(\ln a-\ln R)-1+\nu]\}}{2D_0(1+\nu)R^2}. \end{aligned} \quad (2.19)$$

Finally, making use of equations (2.18), (2.16) and the third equation in (2.15) one can express

$$A_{31} = \frac{Pa^2R^2(D_1-D_0)[a^2(1+\nu)-R^2(3+\nu)]}{16D_1\{D_0[a^2(1-\nu)+R^2(1+\nu)]+D_1(\nu-1)(a^2-R^2)\}}. \quad (2.20)$$

Substituting (2.16) – (2.20) in (2.10) and (2.11) one can express the distributions of deflections and bending moments for the elastic stage of deformation. These distributions are shown in Fig. 2.3 for different values of the transverse pressure. It can

be seen from Fig. 2.3 that M_1 and W are continuous for $r \in [0, R]$ whereas the hoop moment M_2 is discontinuous at $r = a$ (Fig. 2.3 corresponds to the case $a = 0.5R$; $R = 1$ m).

2.4 Optimization of stepped elastic circular plate

The problem posed above consists in the minimization of the maximal deflection $W_0 = W(0)$ under the condition that the material volume of the plate is constrained.

The deflection $W(0)$ is a function of design parameters; $W_0 = W_0(a_1, \dots, a_n, h_0, \dots, h_n)$. The constrained minimum of W_0 can be defined employing the extended function

$$J_* = W_0(a_1, \dots, a_n, h_0, \dots, h_n) + \lambda(V - V_0) \quad (2.21)$$

where λ is an unknown Lagrangian multiplier.

The necessary optimality conditions of (2.21) can be written as

$$\begin{aligned} \frac{\partial J_*}{\partial a_i} &= 0; & i &= 1, \dots, n \\ \frac{\partial J_*}{\partial h_j} &= 0; & j &= 0, \dots, n \\ \frac{\partial J_*}{\partial \lambda} &= 0. \end{aligned} \quad (2.22)$$

The system (2.22) consists of $2n + 2$ algebraic equations. These admit to determine $2n + 2$ unknown parameters a_i ($i = 1, \dots, n$), h_j ($j = 0, \dots, n$) and λ for each given volume of the plate V_0 .

The efficiency of the design established above (see Table 2.1) is assessed by the ratio

$$e = \frac{W_0}{W_*}, \quad (2.23)$$

W_* being the deflection of the plate of constant thickness $h_* = V_0 h_0$. The values of the coefficient e are accommodated in Table 2.1 together with optimal values $\alpha = \frac{a}{R}$ and $\gamma = \frac{h_1}{h_0}$. In the first column of Table 2.1 the values of the plate volume $V_0 = \frac{V}{\pi}$ are presented. It reveals from Table 2.1 that for smaller values of V_0 the deflection of the reference plate of constant thickness W_* is larger, as might be expected. If V_0 increases then the optimal value of α also increases. The maximal deflection of the optimized plate decreases if the quantity V_0 increases. One can see from the Table 2.1 that for $V_0 = 0.45$ the optimal values of parameters α and γ are approximately $\alpha = 0.4904$, $\gamma = 0.2759$. In this case the maximal deflection of the stepped plate is 15% less than that of the reference plate of constant thickness.

Table 2.1: Optimal values of parameters.

V_0	α	γ	$W_o \cdot 10^6$	$W_* \cdot 10^6$	e
0.45	0.490363	0.275881	4.38282	5.15278	0.85057
0.50	0.549226	0.284027	3.86136	4.6375	0.83264
0.55	0.605052	0.290123	3.46249	4.21591	0.82129
0.60	0.658147	0.294337	3.15763	3.86458	0.81707
0.65	0.708775	0.296677	2.92554	3.56731	0.82010
0.70	0.757151	0.296968	2.74998	3.3125	0.83018
0.75	0.803436	0.294764	2.61823	3.09167	0.84687
0.80	0.847728	0.289161	2.5201	2.89844	0.86947
0.85	0.890033	0.278294	2.44735	2.72794	0.89714
0.90	0.930189	0.257875	2.39317	2.57639	0.92888
0.95	0.967623	0.21513	2.35191	2.44079	0.96358
0.99	0.99431	0.118785	2.32495	2.34217	0.99265

2.5 Solution of governing equations in plastic regions in the case of diamond yield condition

In a plastic region one has to satisfy the equations (2.6), (2.7) and the equilibrium equations (1.1). Integrating (2.7) one easily obtains

$$W = Ar^2 + B, \quad (2.24)$$

A and B being arbitrary constants. For determination of stress components one can use equations (1.1) and (2.6). The second equation in system (1.1) gives after integration

$$Q = -\frac{1}{r} \int P(r)rdr + C_1. \quad (2.25)$$

It is worthwhile to mention that the shear force must be continuous for $r \in [0, R]$. However, the function $P = P(r)$ can be discontinuous. In the particular case when $P(r) = \text{const}$ instead of (2.25) one has for $r \in [0, R]$

$$Q = -\frac{Pr}{2} \quad (2.26)$$

where the symmetry condition $Q(0) = 0$ has taken into account.

Substitution of equations (2.6) and (2.25) in (1.1) leads to linear differential equation

$$\frac{dM_1}{dr} + \frac{2M_1}{r} = \frac{M_{0j}}{r} - \frac{1}{r} \int Prdr + C_1 \quad (2.27)$$

for $r \in (0, a)$. In order to find the general solution for equation (2.27) let us first consider the corresponding homogeneous equation. Evidently, the general solution of it has the form

$$M_h = \frac{C}{r^2}. \quad (2.28)$$

The method of variation of the constant in equations (2.27), (2.28) yields

$$C(r) = \frac{M_{00}r^2}{2} - \int \left(r \cdot \frac{Pr^2}{2} \right) dr + \frac{C_1r^3}{3} + C_2. \quad (2.29)$$

Thus the radial bending moment in a plastic region $(0, a)$ is defined as

$$M_1 = \frac{M_{00}}{2} - \frac{1}{r^2} \int \frac{Pr^3}{2} dr + \frac{C_1r}{3} + \frac{C_2}{r^2}. \quad (2.30)$$

Integration constants C_1, C_2 in equation (2.30) can be determined using the continuity requirements of M_1 at the boundary of a plastic zone. If, for instance, the plastic region is located near the center of the plate for $r \in [0, \eta R]$ where $\eta < 1$ then evidently $C_1 = C_2 = 0$ because otherwise the moment M_1 is not limited. Thus in this case

$$M_1 = \frac{M_{00}}{2} - \frac{Pr^2}{8}. \quad (2.31)$$

Here M_{00} denotes the limit moment for the portion of the plate with thickness h_0 . The circumferential moment can be found according to equations (2.6), (2.31) as

$$M_2 = \frac{M_{00}}{2} + \frac{Pr^2}{8}. \quad (2.32)$$

2.6 Elastic plastic solution in the case of diamond condition

During the subsequent increasing the load intensity the plate is divided into elastic and plastic regions. Plastic deformations occur in the central part of the plate with radius y . However, the outward part of the plate remains elastic. The deflection in elastic region is defined by equation (2.10) and moments by equations (2.11). Here one has to take $j = 0$, if $r \in [y, R]$.

In the central plastic region the stress profile lies on the yield curve. Principal moments are defined by equations (2.31), (2.32).

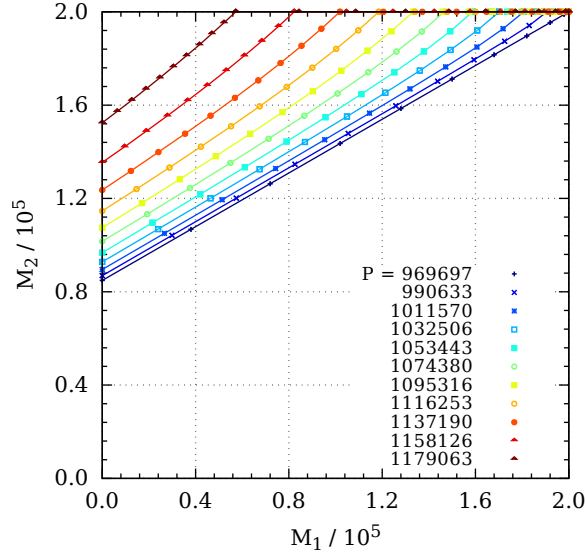


Figure 2.4: Stress profile (Tresca condition).

For determination of the unknown constants A , B , A_{20} , A_{30} , A_{40} and y , one can use the boundary conditions (2.13) and the continuity conditions at $r = y$

$$\begin{aligned}
 [W(y)] &= 0, \\
 [M_1(y)] &= 0, \\
 \left[\frac{dW(y)}{dr} \right] &= 0, \\
 [M_2(y)] &= 0.
 \end{aligned} \tag{2.33}$$

Making use of the second and forth continuity condition in system (2.33) one can determine the constants

$$A_{20} = -\frac{Py^2(\nu + 1) + 4M_0}{16D_0(1 + \nu)} \tag{2.34}$$

and

$$A_{30} = \frac{Py^4(\nu + 1)}{16D_0(\nu - 1)}. \tag{2.35}$$

From the third continuity condition in system (2.33) one can express the constant

$$A = \frac{Py^2(\nu + 1) - 4M_0}{16D_0(\nu^2 - 1)}. \tag{2.36}$$

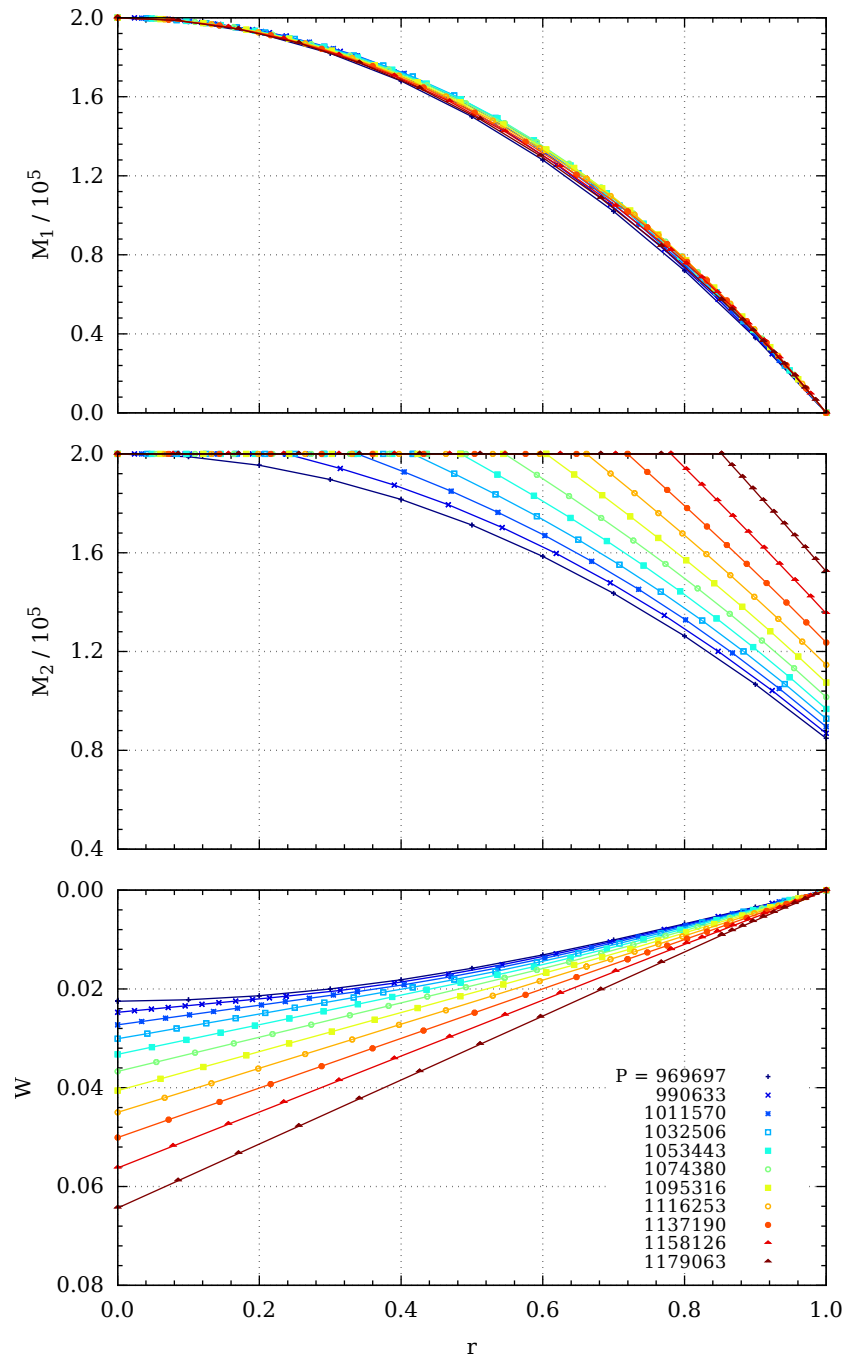


Figure 2.5: Bending moments and deflections (Tresca condition).

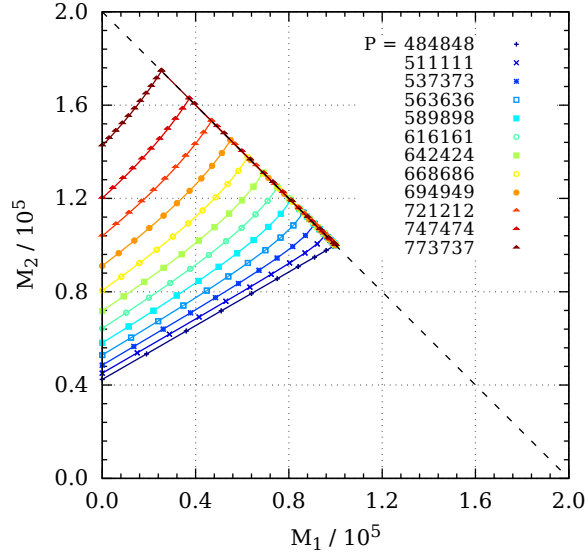


Figure 2.6: Stress profile (diamond condition).

For determining the constant one can use the second boundary condition in equations (2.13). This results in

$$A_{40} = \frac{1}{64D_0(\nu^2 - 1)} \{4R^2(\nu - 1)[Py^2(\nu + 1) + 4M_0] - 4(\nu + 1)^2 y^4 P \ln R - PR^4(\nu^2 - 1)\} \quad (2.37)$$

Making use of the first continuity condition in system (2.33) one can find the constant

$$B = \frac{Py^4[(\nu + 1) \ln y - \nu]}{16D_0(\nu - 1)} + \frac{1}{64D_0(\nu^2 - 1)} \{4R^2(\nu - 1) \cdot [Py^2(\nu + 1) + 4M_0] - 4(\nu + 1)^2 y^4 P \ln R + P(\nu^2 - 1)(y^4 - R^4)\}. \quad (2.38)$$

Finally, for expressing the constant y one can substitute (2.34) and (2.35) in the first boundary condition in equations (2.13). This leads to the equation

$$P(\nu + 1)y^4 - 2(\nu + 1)PR^2y^2 - 8M_0R^2 + PR^4(3 + \nu) = 0. \quad (2.39)$$

It is easy to solve the biquadratic equation (2.39) and get the constant

$$y = \sqrt{R^2 \pm \sqrt{\frac{-2R^2(PR^2 - 4M_0)}{P(\nu + 1)}}}. \quad (2.40)$$

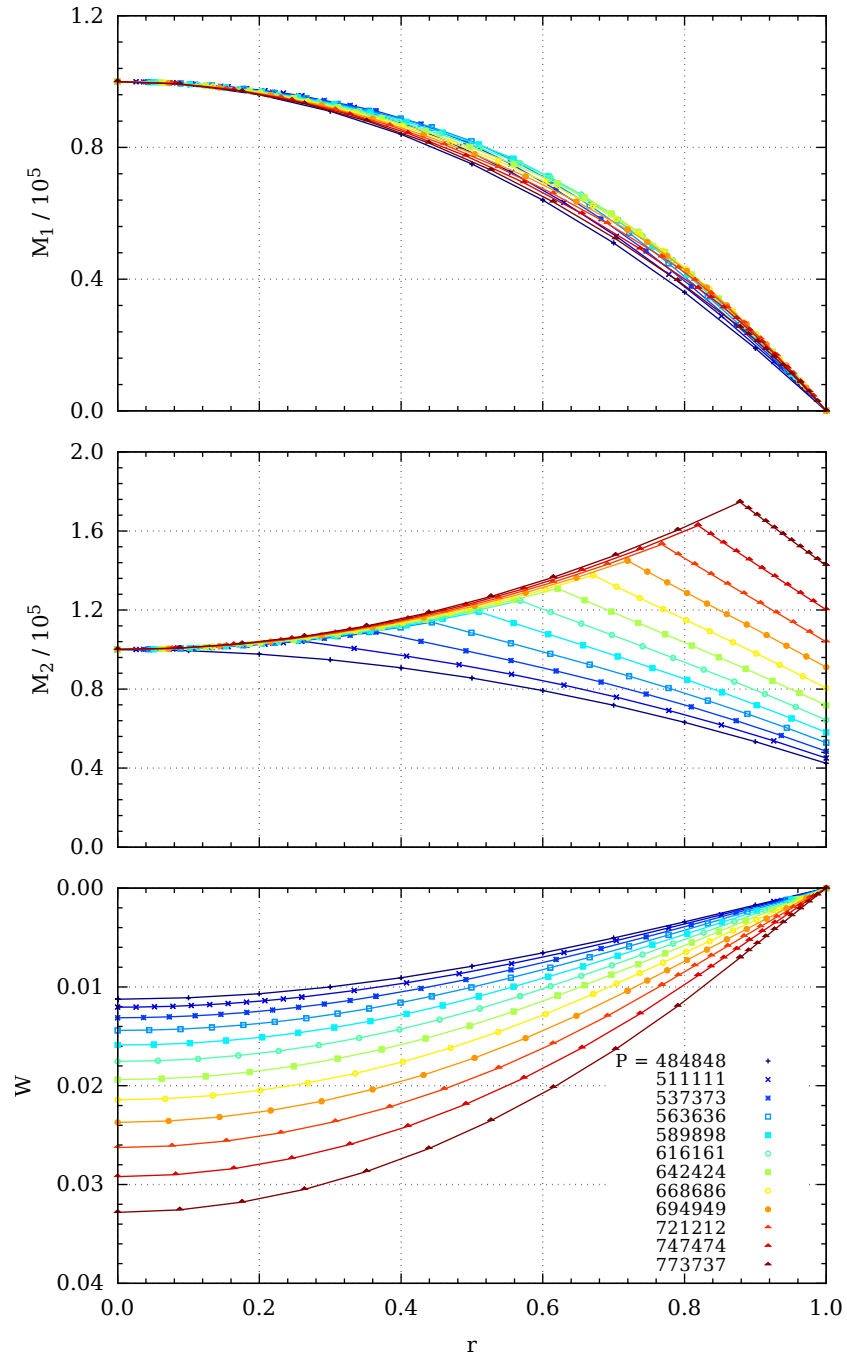


Figure 2.7: Bending moments and deflections(diamond condition).

Evidently, only the choice of the sign "minus" in (2.40) leads to a realistic solution.

The results of calculations are presented in Fig. 2.6 and Fig. 2.7 for different values of the transverse pressure. Comparing the results presented in Fig. 2.5 and Fig. 2.7 one can see that the plate made of a Tresca material is stronger than the plate corresponding to the diamond yield condition. One can make the same conclusion observing the Fig. 2.8. Although the hoop moment distributions are quite different in these cases the shapes of radial moments are similar. However, the moments are of different size in these cases.

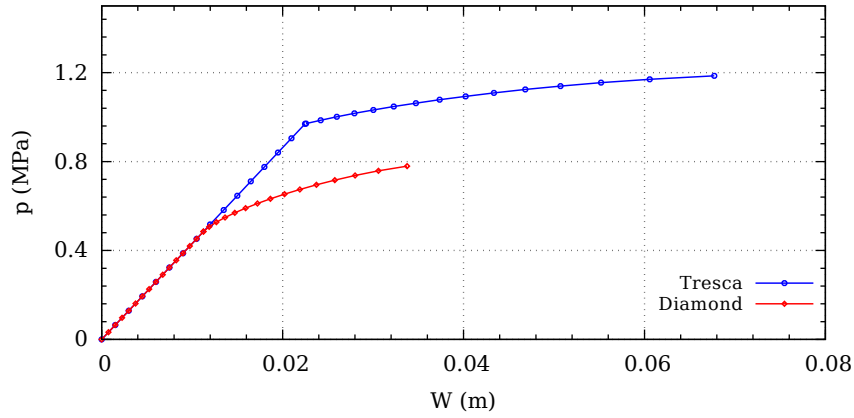


Figure 2.8: Load-deflection relations of circular plates.

2.7 Elastic plastic bending of an annular plate

Elastic plastic deformations of an annular plate with radii a and R will be studied (see Fig. 2.9). The plate is subjected to the distributed transverse loading of intensity $P = P(r)$. Assume that the outer edge is simply supported whereas the inner edge of the plate is completely free. Therefore, at the outer edge the transverse deflection W and the radial bending moment M_1 must vanish. The radial bending moment is zero at the inner edge, as well. Thus, the boundary conditions are at the outer edge

$$\begin{aligned} W(R) &= 0, \\ M_1(R) &= 0 \end{aligned} \tag{2.41}$$

and at the inner edge

$$\begin{aligned} M_1(a) &= 0, \\ Q(a) &= 0 \end{aligned} \tag{2.42}$$

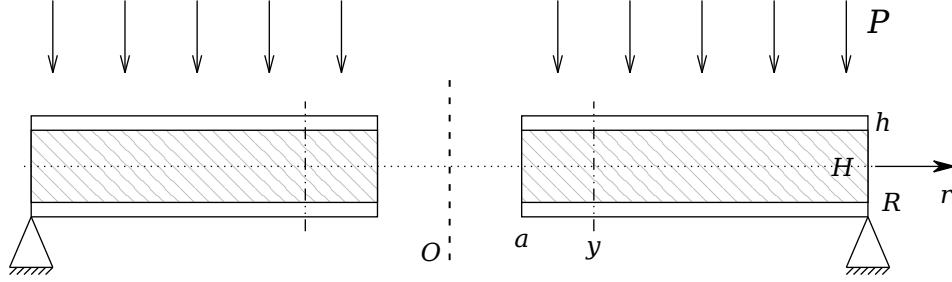


Figure 2.9: Cross-section of the annular plate.

where Q is the shear force. Note that the hoop moment M_2 can take arbitrary values at the both edges. For the sake of simplicity it is assumed that the cross sections of the plate are of sandwich type. The aim of this section is to determine the distributions of bending moments M_1 , M_2 and the transverse deflection W for each value of the transverse load. It is expected that in the range of low loadings the plate is fully elastic and with the subsequent increase of the load level elastic plastic deformations occur.

2.7.1 Elastic stage of deformation

In the case of low stress level the plate remains elastic and the Hooke's law holds good. The stresses are coupled with external loads by the equilibrium equations (1.1). In the case $N_1 = N_2 = 0$ equations (1.1) can be presented as

$$\frac{d}{dr}(rM_1) - M_2 - rQ = 0 \quad (2.43)$$

and

$$\frac{d}{dr}(rQ) = -P(r)r. \quad (2.44)$$

In the present case it is assumed that $P(r) = \text{const}$. This admits to integrate the last equation. The solution of 2.44 satisfying the boundary condition (2.42) can be presented as

$$Q = -\frac{P(r^2 - a^2)}{2r}. \quad (2.45)$$

It can be rechecked that the solution of (1.14) satisfying (2.41) is

$$W = \frac{P(r^4 - R^4)}{64D} + A_1(r^2 \ln r - R^2 \ln R) + A_2(r^2 - R^2) + A_3 \ln \frac{r}{R}, \quad (2.46)$$

$A_1 - A_3$ being arbitrary constants. However, calculating the shear force Q from (2.43) making use of (1.11), (1.8), (2.46) and comparing with (2.45) one can see that

$$A_1 = -\frac{Pa^2}{8D} \quad (2.47)$$

Substituting (2.46) in (1.11) one can determine the bending moments

$$M_1 = -\frac{Pr^2(3+\nu)}{16} + \frac{Pa^2[2(1+\nu)\ln r + 3+\nu]}{8} - D \left[2A_2(1+\nu) + \frac{A_3(\nu-1)}{r^2} \right] \quad (2.48)$$

and

$$M_2 = -\frac{Pr^2(1+3\nu)}{16} + \frac{Pa^2[2(1+\nu)\ln r + 1+3\nu]}{8} - D \left[2A_2(1+\nu) + \frac{A_3(1-\nu)}{r^2} \right]. \quad (2.49)$$

Applying the boundary conditions (2.41), (2.42) to (2.48) one can define

$$A_2 = \frac{P}{32D(1+\nu)(R^2-a^2)} \cdot \{ (3+\nu)(a^4-R^4) + 2a^2[(R^2-a^2)[2(1+\nu)\ln a + 3+\nu] - 2R^2(1+\nu)(\ln a - \ln R)] \}, \quad (2.50)$$

$$A_3 = \frac{Pa^2R^2[(3+\nu)(R^2-a^2) + 4a^2(1+\nu)(\ln a - \ln R)]}{16D(\nu-1)(R^2-a^2)}. \quad (2.51)$$

2.7.2 Elastic plastic stage of deformation

The plate remains pure elastic until the stress profile lies entirely inside the Tresca's yield hexagon (Fig. 1.3). During the elastic stage the inequalities

$$|M_1| \leq M_0, \quad |M_2| \leq M_0, \quad |M_1 - M_2| \leq M_0 \quad (2.52)$$

are satisfied as strict inequalities. The elastic plastic stage begins at the load level $P = P_1$ when the stress profile reaches to the yield locus. The analysis shows that the stress profile reaches to the side $M_2 = M_0$ of the yield hexagon. The maximum of the hoop moment is achieved at the inner edge of the plate and thus, the quantity P_1 can be calculated from the equation $M_2(a) = M_0$. Therefore,

$$P_1 = \frac{8M_0(R^2-a^2)}{R^4(3+\nu) + 4a^2R^2[(1+\nu)(\ln a - \ln R) - 1] + a^4(1-\nu)}. \quad (2.53)$$

During the elastic plastic stage the plate is divided into two parts. In the inner part (a, y) plastic deformations take place whereas the region (y, R) remains elastic. Let us consider the regions separately.

Plastic region, $r \in (a, y)$

For $r \in (a, y)$ the stress profile lies on the side $M_2 = M_0$ of the yield hexagon (Fig. 1.3). Substituting $M_2 = M_0$ in (2.43) and taking (2.45) into account one obtains for $r \in (a, y)$

$$rM_1 - rM_0 + \frac{P}{2} \left(\frac{r^3}{3} - a^2r \right) = C, \quad (2.54)$$

C being an arbitrary constant. Due to the boundary condition (2.42)

$$C = -aM_0 - \frac{Pa^3}{3}. \quad (2.55)$$

Thus (2.54) can be put into the form

$$M_1 = \frac{r-a}{r} \left[M_0 - \frac{P(r^2 + ar - 2a^2)}{6} \right]. \quad (2.56)$$

It is worthwhile to emphasize that (2.56) holds good for $r \in (a, y)$. The deflection W is expressed by the formula (1.23).

Elastic region, $r \in (y, R)$

In the elastic region of the plate the relations (2.46) – (2.49) hold good. However, (2.50) and (2.51) are not valid. For determination of unknown constants A_2 , A_3 and A , B one can use the boundary conditions at $r = R$, also the continuity requirements of quantities W , $\frac{dW}{dr}$, M_1 , M_2 at $r = y$. The requirements $M_1(R) = 0$ and $M_2(y) = M_0$ give with the help of (2.48), (2.49)

$$\begin{aligned} & -\frac{PR^2(3+\nu)}{16} + \frac{Pa^2[2(1+\nu)\ln R + 3+\nu]}{8} \\ & -D \left[2A_2(1+\nu) + \frac{A_3(\nu-1)}{R^2} \right] = 0 \end{aligned} \quad (2.57)$$

and

$$\begin{aligned} & -\frac{Py^2(1+3\nu)}{16} + \frac{Pa^2[2(1+\nu)\ln y + 1+3\nu]}{8} \\ & -D \left[2A_2(1+\nu) + \frac{A_3(1-\nu)}{y^2} \right] = M_0. \end{aligned} \quad (2.58)$$

The continuity of the radial bending moment at $r = y$ with (2.48) and (2.56) furnishes the relation

$$\begin{aligned} & \frac{Pa^2[2(1+\nu)\ln y + 3+\nu]}{8} - D \left[2A_2(1+\nu) + \frac{A_3(\nu-1)}{y^2} \right] \\ & -\frac{Py^2(3+\nu)}{16} - \frac{y-a}{y} \left[M_0 - \frac{P(y^2 + ay - 2a^2)}{6} \right] = 0. \end{aligned} \quad (2.59)$$

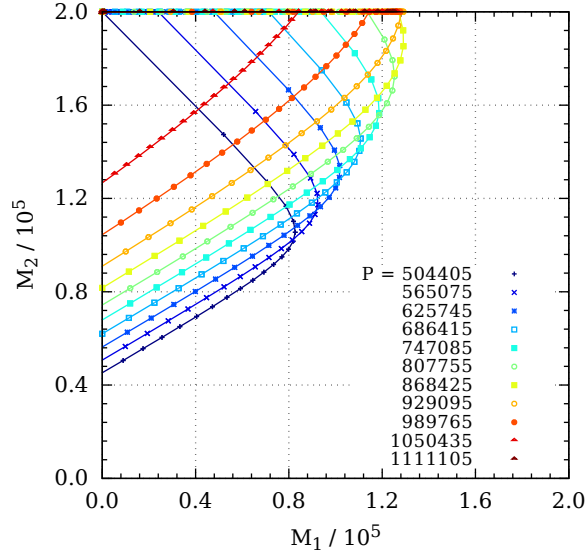


Figure 2.10: Stress profile ($a = 0.1R$).

Due to the continuity of the deflection and its slope one has making use of (2.46) and (1.23)

$$A = \frac{Py^3}{16D} - \frac{Pa^2y(2\ln y) + 1}{8D} + 2A_2y + \frac{A_3}{y} \quad (2.60)$$

and

$$\begin{aligned} & -\frac{P(y^4 - R^4)}{64D} - \frac{Pa^2(y^2 \ln y - R^2 \ln R)}{8D} + A_2(y^2 - R^2) \\ & + A_3 \ln \frac{y}{R} - Ay - B = 0. \end{aligned} \quad (2.61)$$

From (2.59), (2.58) one can easily define

$$\begin{aligned} A_2 = & \frac{1}{48D(1+\nu)y} \cdot \{-Py^3(1+3\nu) + 6Pa^2y[(\nu+1)\ln y + \nu] \\ & + 12M_0(a-2y) + 4Pa^3\} \end{aligned} \quad (2.62)$$

and

$$A_3 = \frac{y[Py^3(1+3\nu) - 6Pa^2y(1+\nu) + 8a(3M_0 + Pa^2)]}{48D(\nu-1)}. \quad (2.63)$$

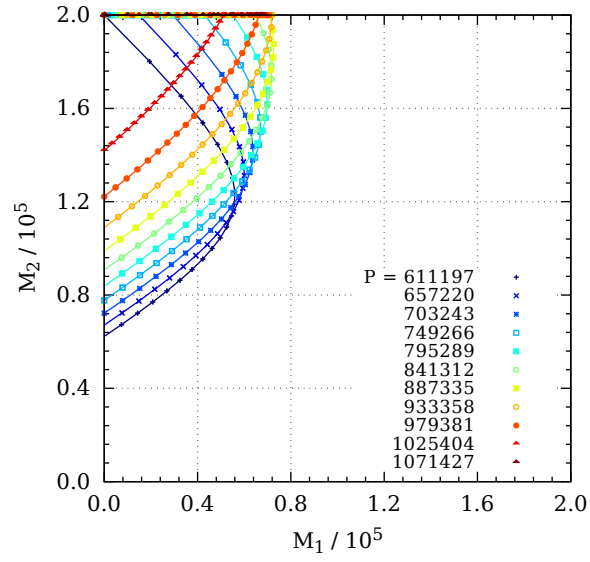


Figure 2.11: Stress profile ($a = 0.3R$).

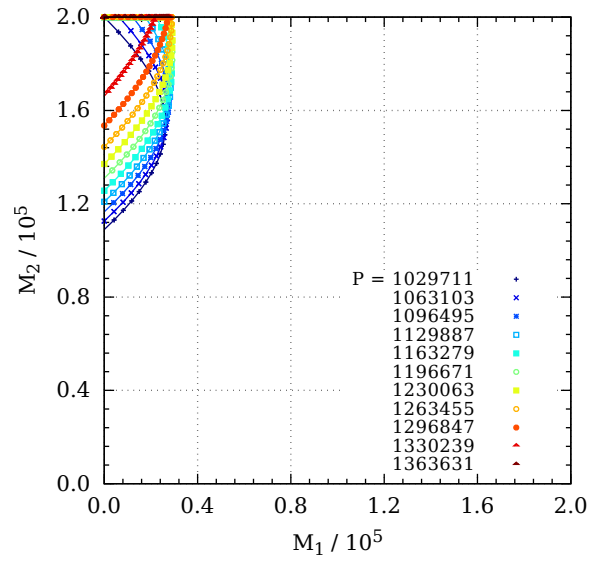


Figure 2.12: Stress profile ($a = 0.6R$).

The equation (2.61) admits to define

$$B = \frac{P[-3y^4 + 8a^2y^2(\ln y + 1) - R^2(R^2 - 8a^2 \ln R)]}{64D} - (y^2 + R^2)A_2 + \left(\ln \frac{y}{R} - 1\right) A_3. \quad (2.64)$$

Substituting A_2 , A_3 from (2.62), (2.63) in (2.57) one obtains the equation

$$\begin{aligned} & -\frac{PR^2(3+\nu)}{16D} + \frac{Pa^2[2(1+\nu)\ln R + 3+\nu]}{8D} + \frac{1}{24Dy} \\ & \cdot \{-Py^3(1+3\nu) + 6Pa^2y[(\nu+1)\ln y + \nu] + 12M_0(a-2y) + 4Pa^3\} \\ & + \frac{y[Py^3(1+3\nu) - 6Pa^2y(1+\nu) + 8a(3M_0 + Pa^2)]}{48DR^2} = 0. \end{aligned} \quad (2.65)$$

The equation (2.65) serves for determination of the quantity y for fixed load intensity P .

2.7.3 Numerical results

The equation (2.65) is solved numerically making use of the computer code *Mathematica*. The results of calculations are presented in Table 2.2 and Fig. 2.10 – Fig. 2.15 for the plate with radius $R = 1$ m, $h = 0.01$ m and $H = 0.05$ m. Calculations carried out showed that the plastic region expands with the growth of the load intensity (Table 2.2). The plate reaches to the fully plastic stage when $P = 1071428, 57$ Pa.

Table 2.2: Radius of the plastic zone.

P	611197	657220	703243	749266	795289	841312
y	0.3	0.3245	0.3536	0.3884	0.4304	0.4808
P	887335	933358	979381	1025404	1071427	
y	0.5409	0.6115	0.6939	0.7941	0.9989	

The distributions of bending moments of the plate are depicted in Fig. 2.10 – Fig. 2.12 for different values of the load intensity. It can be seen from Fig. 2.10 – Fig. 2.12 that the distributions of bending moments are statically admissible since at each $r \in [a, R]$ corresponding values of M_1 , M_2 , are such that the point either lies on the side of the yield hexagon or is located inside the hexagon. The elastic

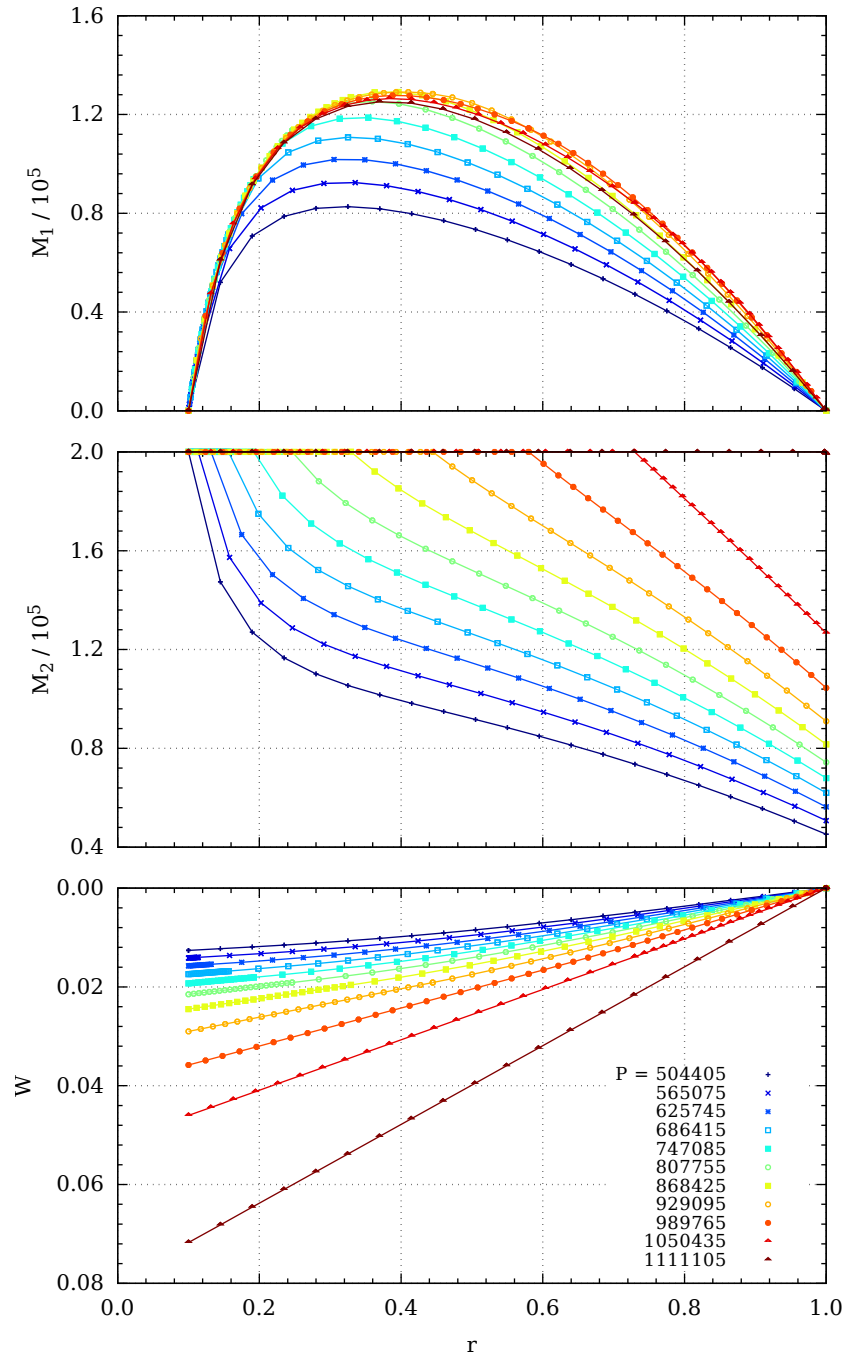


Figure 2.13: Bending moments and deflections ($a = 0.1R$).

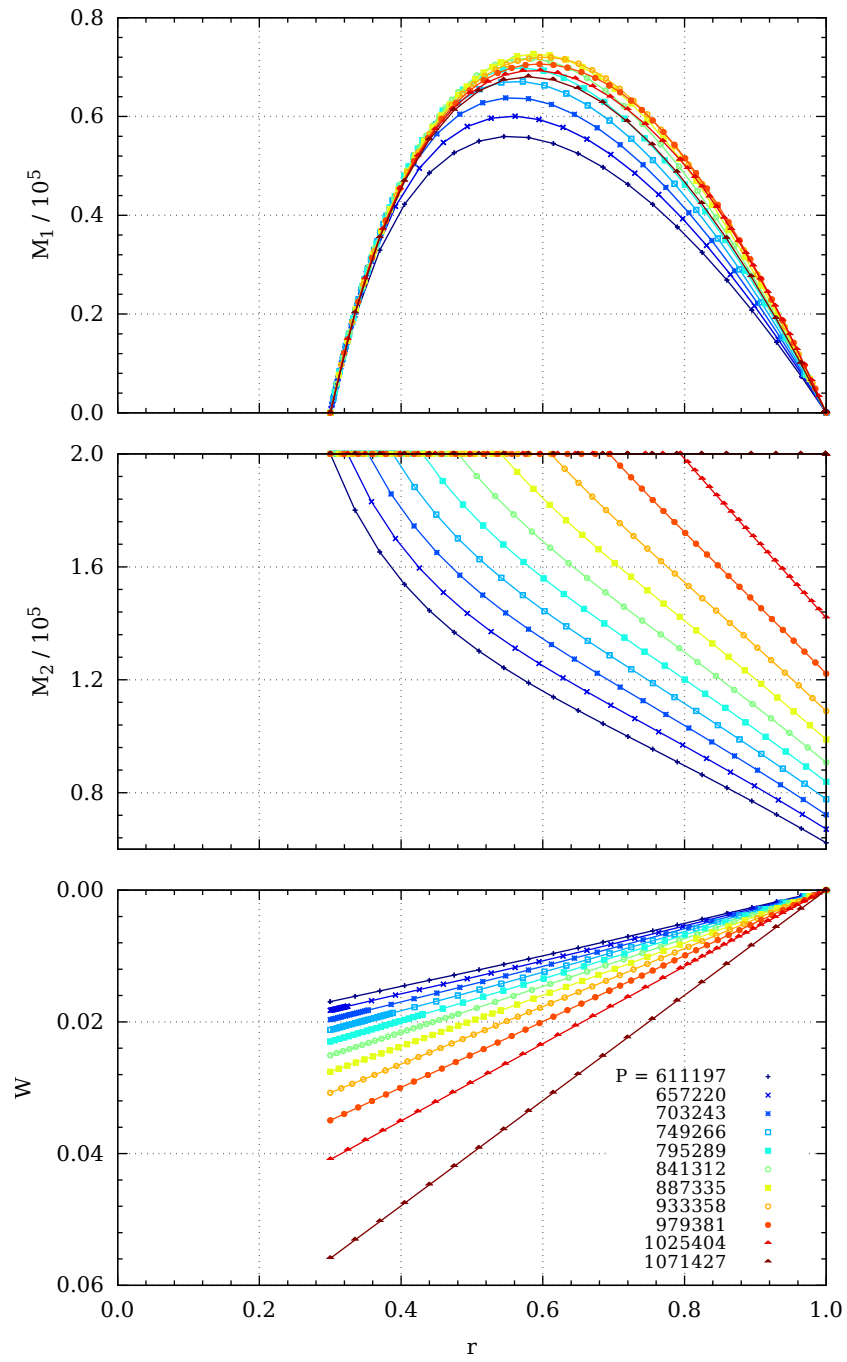


Figure 2.14: Bending moments and deflections ($a = 0.3R$).

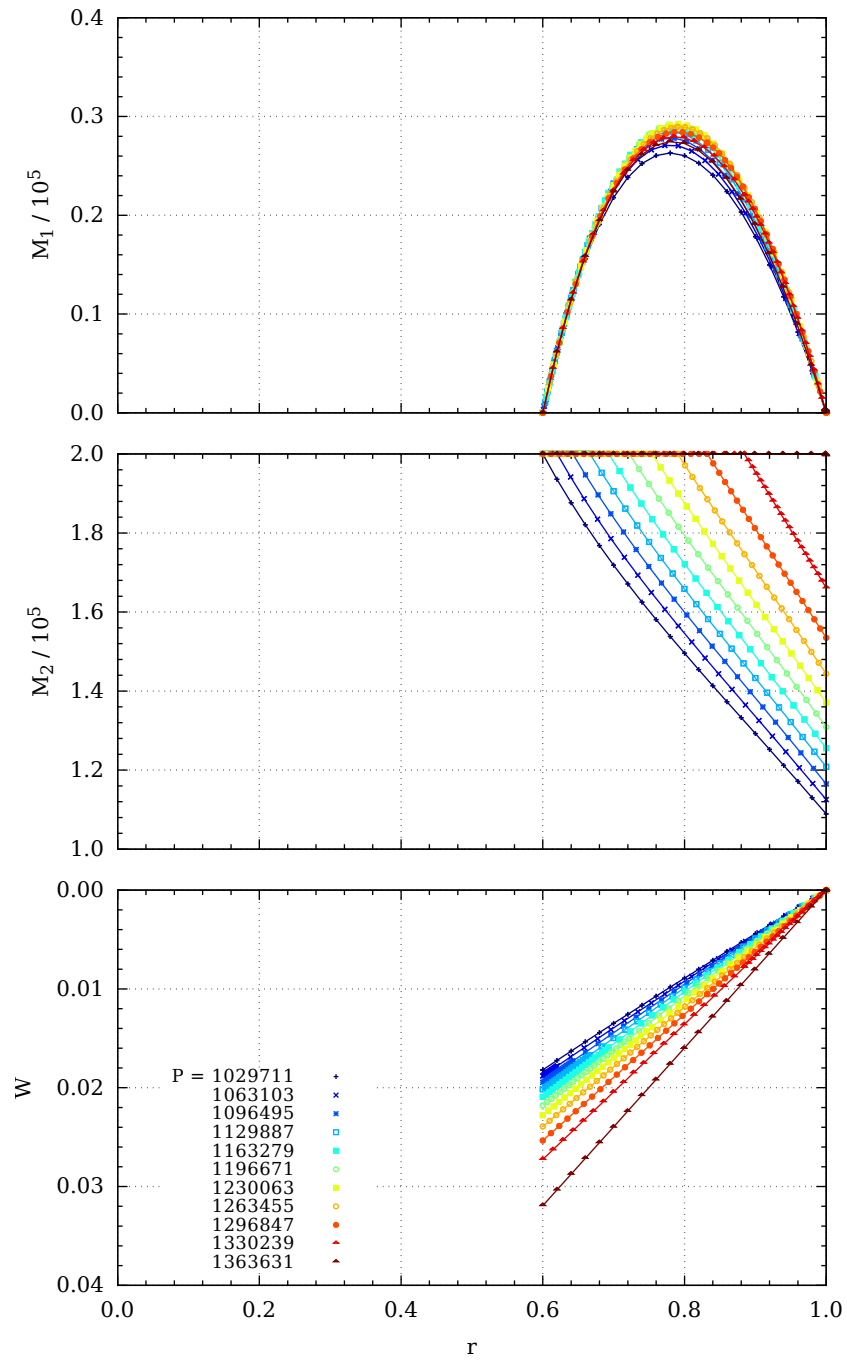


Figure 2.15: Bending moments and deflections ($a = 0.6R$).

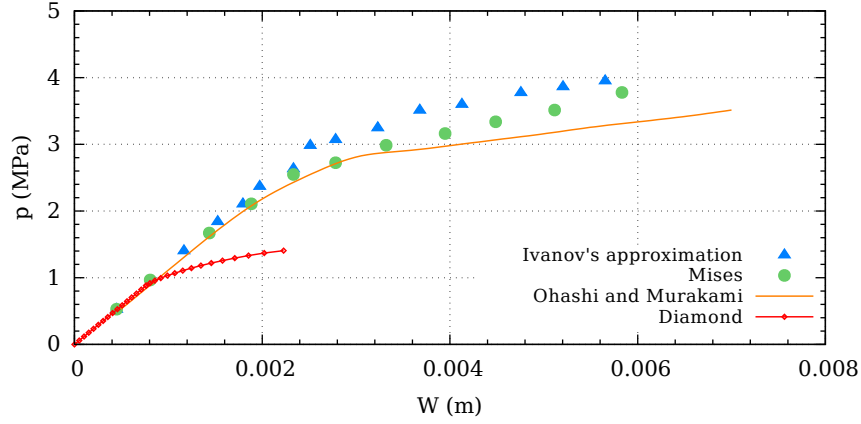


Figure 2.16: Load-deflection relations of circular plates.

plastic stage of deformation is completed at the loading level where the whole plate is plastic. This situation corresponds to

$$P = P_2 = \frac{6M_0(R - a)}{R^3 + 2a^3 - 3a^2R} \quad (2.66)$$

and $y = R$.

It is worthwhile to mention that the stress profiles (see Fig. 2.10 – Fig. 2.12, also Fig. 2.4, Fig. 2.6) do not present the bending moments as functions of the current radius. However, these figures admit to prove the statical admissibility of the current solution.

The transverse deflections and the bending moments are depicted in Fig. 2.13 – Fig. 2.15 for the plate of constant thickness with different inner radii. It can be seen from Fig. 2.13 – Fig. 2.15 that the distributions of radial moments depend slightly on the load intensity. However, the hoop moment distribution is completely different in the elastic and elastic plastic stages of deformation. As it was shown above $M_2 = M_0$ in the plastic region. If the load intensity P tends to the value P_2 then the hoop moment distribution tends to the value $M_2 \equiv M_0$ for $r \in [a, R]$. The load-deflection relations of plates made of different materials are presented in [97]. The results of [97] are compared with the current solution in Fig. 2.16. It can be seen from Fig. 2.16 that the curve corresponding to the material obeying the diamond yield condition is lower of these corresponding to the Mises yield condition and its non-linear approximations. The matter that the curve corresponding to the diamond yield condition is lower than the others coincides with our expectation because the diamond on the plane of principal moments can be considered as an inscribed polygon with respect to the Mises ellipse.

CHAPTER 3

ELASTIC PLASTIC BENDING OF STEPPED ANNULAR PLATES

3.1 The model of the plate

Let us consider the axisymmetric bending of an annular plate subjected to the transverse pressure of intensity P . Assume that the internal edge of the plate of radius a is clamped whereas the external edge of radius R is absolutely free. The plate under consideration has sandwich type cross section (see Fig. 3.1).

The behaviour of the plate will be prescribed with the first order bending theory of thin plates corresponding to small deformations and small displacements (see [82, 102]). The stress state is defined by bending moments M_1 , M_2 in the radial and circumferential direction, respectively. The latters are coupled with principal stresses σ_1 , σ_2 by (1.9). Due to the symmetry the moment $M_{12} = 0$. Note that the membrane forces will be neglected according to current approach. The third generalized stress component to be taken into account is the shear force Q but this component does not contribute to the strain energy when the bending theory is used. Moreover, the shear force, although appearing in the equilibrium conditions, can be eliminated from the equilibrium equations and thus does not involve in the set of governing equations.

It is assumed that the stress strain state induced by the axisymmetric transverse pressure is axisymmetric at each stage of the pressure. Thus the stress and strain components are defined at each point of the plate by the current radius and the given pressure level.

Material of the plate is assumed to be an ideal elastic plastic material obeying the square yield condition in the plastic (inelastic) stage of deformation.

The aim of the chapter is to determine the transverse deflection as well as bending moments distributions in the elastic and subsequent inelastic stages of deformation for given transverse pressure levels.

3.2 Basic equations and concepts

It is well known that in the case of lower values of the pressure loading the plate is pure elastic. The elastic behaviour of the material can be prescribed with Hooke's

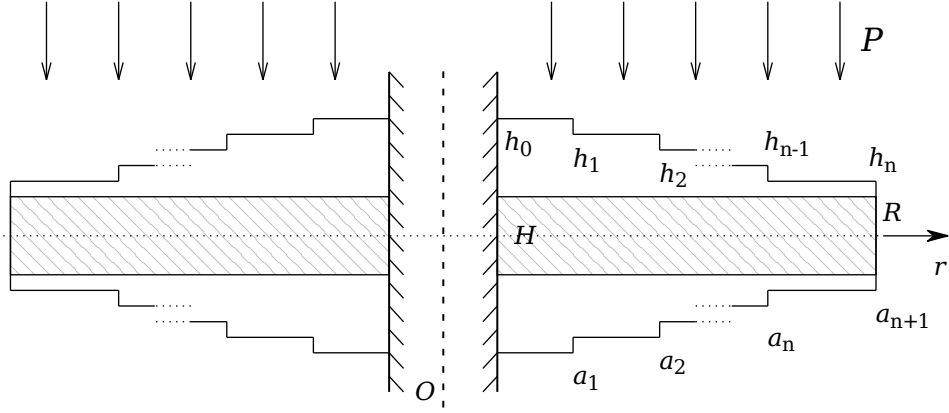


Figure 3.1: Cross-section of the plate.

law. The latter is to be presented in the generalized form as

$$\begin{aligned} M_1 &= D_j(\varkappa_1 + \nu \varkappa_2), \\ M_2 &= D_j(\varkappa_2 + \nu \varkappa_1) \end{aligned} \quad (3.1)$$

where $j = 0, \dots, n$. In the case of a sandwich plate D_j is presented by the formula (2.9).

During the subsequent quasistatic increasing the external loading constitutive equations (3.1) hold good until the elastic limit is exhausted at an unknown point of the plate. In the case of a full plate and of the pressure of constant intensity the yield limit is reached at first at the center of the plate. After that the plate is subdivided into elastic and plastic regions, respectively. Let these regions be S_e and S_p , respectively. Since we are studying the plate of sandwich type and the carrying layers are thin no elastic plastic state of deformations occurs.

Assume that the material of the plate obeys the square yield condition and associated flow rule (Fig. 3.2). Thus for $r \in S_p$ the stress state of the point is such that the point $(M_1(r), M_2(r))$ lies on a side of the square (Fig. 3.2). It means that

$$|M_1| \leq M_{0j}, \quad |M_2| \leq M_{0j} \quad (3.2)$$

for $r \in (a_j, a_{j+1})$, $j = 0, \dots, n$. Quantity M_{0j} stands for the yield moment corresponding to the thickness h_j . It can be easily stated that M_{0j} is presented in form (2.5).

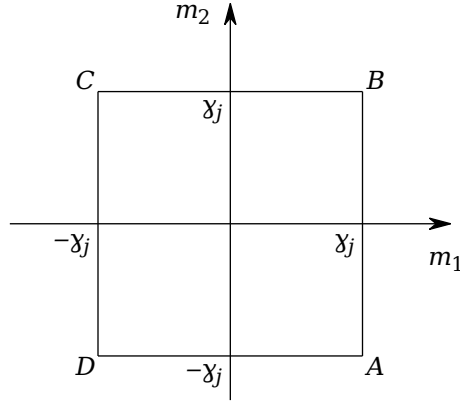


Figure 3.2: Square yield condition.

Evidently, at the boundary of the plate requirements

$$\begin{aligned} M_1(R) &= 0, \\ Q(R) &= 0 \end{aligned} \tag{3.3}$$

and

$$W(a) = 0 \tag{3.4}$$

must be satisfied at each loading level.

Let us consider the governing equations separately in elastic and plastic regions, respectively. In elastic regions the stress strain state is determined according to (1.1) and (3.1). The substitution of (3.1) and (1.8) in (1.1) easily leads to the equation (2.8).

In the following it is reasonable to use non-dimensional quantities

$$\begin{aligned} \rho &= \frac{r}{R}, & m_1 &= \frac{M_1}{M_*}, & m_2 &= \frac{M_2}{M_*}, & q &= \frac{RQ}{M_*}, \\ \alpha &= \frac{a}{R}, & \alpha_j &= \frac{a_j}{R}, & p &= \frac{PR^2}{M_*}, & w &= \frac{W}{H}, \\ \gamma_j &= \frac{h_j}{h_*}, & d_j &= \frac{EH^2h_j}{2(1-\nu^2)\sigma_0R^2h_*} \end{aligned} \tag{3.5}$$

where $M_* = \sigma_0 h_* H$ is the yield moment of a reference plate of constant thickness h_* .

Making use of variables (3.5) one can present the equilibrium equations (1.1) as

$$((\rho m_1)' - m_2)' + p\rho = 0 \quad (3.6)$$

where primes denote the differentiation with respect to the non-dimensional radius ρ .

3.3 General solutions in elastic and plastic regions

Let us denote an elastic region (a_j, a_{j+1}) , $j = 0, \dots, n$ where the thickness of carrying layers is h_j by S_{ej} .

Making use of (2.8) and (3.5) it is easy to recheck that the general solution of (2.8) can be presented as

$$w = A_{1j}\rho^2 \ln \rho + A_{2j}\rho^2 + A_{3j} \ln \rho + A_{4j} + \frac{p\rho^4}{64d_j} \quad (3.7)$$

where $\rho \in S_{ej}$ and $d_j = \frac{D_j H}{M_* R^2}$. Arbitrary constants $A_{1j} - A_{4j}$ will be determined from the boundary requirements and continuity conditions for w , w' , m_1 at the boundaries between elastic and plastic regions.

Non-dimensional bending moments can be determined according to (3.1) and (3.5) as

$$\begin{aligned} m_1 &= -d_j \left(w'' + \frac{\nu}{\rho} w' \right), \\ m_2 &= -d_j \left(\frac{w'}{\rho} + \nu w'' \right) \end{aligned} \quad (3.8)$$

for $\rho \in S_{ej}$. The terms with derivatives w' , w'' in (3.8) can be expressed as

$$w' = A_{1j}(2\rho \ln \rho + \rho) + 2A_{2j}\rho + \frac{A_{3j}}{\rho} + \frac{p\rho^3}{16d_j} \quad (3.9)$$

and

$$w'' = A_{1j}(2 \ln \rho + 3) + 2A_{2j} - \frac{A_{3j}}{\rho^2} + \frac{3p\rho^2}{16d_j} \quad (3.10)$$

for $\rho \in S_{ej}$.

The third stress component besides bending moments is the shear force. It is reasonable to calculate it from the equilibrium equations (1.1) or (3.6). From (3.6), (1.1) one easily obtains the equation

$$(\rho q)' = -p\rho \quad (3.11)$$

which holds good over the entire plate. The solution of this equation which satisfies the boundary condition $q(1) = 0$ is

$$q = -\frac{p}{2} \left(\rho - \frac{1}{\rho} \right) \quad (3.12)$$

for $\rho \in (\alpha, 1)$.

The solution of basic equations in a plastic region S_p depends on the particular yield regime. It appears that in the present case the stress strain state in a plastic region of the plate corresponds to the sides AD or DC of the square yield condition (Fig. 3.2). Let us consider these yield regimes in a greater detail.

In the case of the yield profile CD one has $m_1 = -\gamma_j$ for $\rho \in S_{pj} \subset (a_j, a_{j+1})$. However, it can be rechecked (see [11, 86]) that this regime can not take place at a region of finite length.

If the stress profile lies on the side AB of the yield square (Fig. 3.2) then $m_2 = -\gamma_j$ and after integration of (3.6) with (3.12) one has

$$m_1 = -\gamma_j - \frac{p}{2} \left(\frac{\rho^2}{3} - 1 \right) + \frac{E_j}{\rho} \quad (3.13)$$

for $\rho \in S_{pj}$.

In the case of the yield regime AD one has $\varkappa_1 = 0$ and thus

$$w = A_j \rho + B_j \quad (3.14)$$

for $\rho \in S_{pj}$, where A_j, B_j are arbitrary constants. However, on the side CD of the yield condition (Fig. 3.2) $\varkappa_2 = 0$ and thus $w = \text{const}$. This is one of but not the only reason why the regime CD does not take place in a region of finite length.

3.4 The pure elastic stage of deformation (stage I)

As the intensity of the pressure loading is increased from zero, the entire plate is elastic until the stress profile reaches a side of the yield condition. However, during the elastic stage the stress profile lies inside the square $ABCD$ (Fig. 3.2) where $\gamma = \min \gamma_j$.

During this stage of loading the deflection is defined by (3.7), bending moments and the shear force by (3.8) and (3.12), respectively. For determination of unknown constants one can use the boundary conditions $w'(a) = w(a) = 0$, $m_1(1) = 0$ and the continuity requirements

$$\begin{aligned} [w(\alpha_j)] &= 0, \\ [w'(\alpha_j)] &= 0, \\ [m_1(\alpha_j)] &= 0 \end{aligned} \quad (3.15)$$

for $j = 1, \dots, n$.

It appears that the general solution in the form (3.7) may involve inadequate solutions for particular cases. In order to avoid this one has to check if the shear force in the form (3.12) concides with that following from the first equation of the system (1.1).

Let us denote

$$\rho \bar{q} = (\rho m_1)' - m_2. \quad (3.16)$$

Evidently, $\bar{q} = q$. Thus one has to check if the constraints

$$\bar{q}(\rho) = q(\rho) \quad (3.17)$$

for $\rho \in S_{ej}$ ($j = 1, \dots, n$) with the boundary condition $\bar{q}(1) = 0$ are satisfied. Making use of (3.7) - (3.10) and (3.12) with (3.16) it is easy to show that equalities (3.17) take place if

$$A_{1j} = -\frac{p}{8d_j} \quad (3.18)$$

for $j = 0, \dots, n$.

Thus, for determination of $3n + 3$ unknown constants A_{2j}, A_{3j}, A_{4j} ($j = 0, \dots, n$) one has three boundary conditions and $3n$ continuity conditions (3.15).

Making use of (3.8), (3.9), (3.10) and taking (3.18) into account it is easy to determine the bending moments

$$\begin{aligned} m_1 = & \frac{-p\rho^2(3+\nu)}{16} - \frac{p}{8}[2(1+\nu)\ln\rho + 3+\nu] \\ & -2(1+\nu)A_{2j}d_j + \frac{A_{3j}d_j(1-\nu)}{\rho^2} \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} m_2 = & \frac{-p\rho^2(1+3\nu)}{16} - \frac{p}{8}[2(1+\nu)\ln\rho + 1+3\nu] \\ & -2(1+\nu)A_{2j}d_j + \frac{A_{3j}d_j(1-\nu)}{\rho^2} \end{aligned} \quad (3.20)$$

for $\rho \in (a_j, a_{j+1})$, $j = 0, \dots, n$.

It is interesting to remark that the distribution of the shear force does not depend on the distribution of thicknesses, as it can be seen from (3.12). At the same time other stress components (bending moments) do depend on the thicknesses.

The elastic loading stage completes at the moment when the stress profile reaches the side CD of the yield square. In the case of a plate of constant thickness the plastic yielding happens at first at the internal edge of the plate for $\rho = \alpha$. At the limit stage between the fully elastic stage and inelastic stage for $p = p_0$

$$m_1(\alpha) = -\gamma_0. \quad (3.21)$$

Note that, in principle, the plastic yielding may start elsewhere, as well. If, for instance, the inner annulus is very narrow and the thickness h_0 is large whereas the next annulus has very small thickness then the yield can start from the next annulus. However, these cases will not be studied in the present work.

3.5 Elastic plastic stage with the hinge circle (stage II)

Assume that during this stage of deformation the plastic hinge circle is located at the internal edge of the plate for $\rho = \alpha$. However, due to the hinge the boundary condition $w'(\alpha) = 0$ is no more valid. For determination of unknown constants A_{2j} , A_{3j} , A_{4j} one can use relations (3.15) – (3.17) with boundary conditions $w(\alpha) = 0$, $m_1(1) = 0$ and (3.21). Note that (3.18) remains valid, as well. The latter admits to present the deflection for $\rho \in [\alpha_j, \alpha_{j+1}]$ as

$$w = \frac{p\rho^2(\rho^2 - 8 \ln \rho)}{64d_j} + A_{2j}\rho^2 + A_{3j} \ln \rho + A_{4j}. \quad (3.22)$$

Making use of (3.22), (3.19) and satisfying boundary conditions $w(\alpha) = 0$, $m_1(\alpha) = -\gamma_0$ results in

$$\begin{aligned} A_{20}\alpha^2 + A_{30} \ln \alpha &= -A_{40} - \frac{p\alpha^2(\alpha^2 - 8 \ln \alpha)}{64d_0}, \\ -2A_{20}(1 + \nu)d_0 + \frac{d_0(1 - \nu)A_{30}}{\alpha^2} & \\ = -\gamma_0 + \frac{p\alpha^2(3 + \nu)}{16} + \frac{p[2(1 + \nu) \ln \alpha + 3 + \nu]}{8} & \end{aligned} \quad (3.23)$$

Finally, employing the continuity conditions (3.15) with boundary conditions (3.23) and $m_1(1) = 0$ admits to determine unknown constants A_{2j} , A_{3j} , A_{4j} for each $j = 0, \dots, n$.

This stage of deformation will be completed when the stress profile at $\rho = \alpha$ reaches the point D (Fig. 3.2) so that $m_2(\alpha) = -\gamma_0$. Let the corresponding value of the external load intensity be p_1 .

3.6 The elastic plastic stage with a plastic region of finite length (stage III)

It is reasonable to assume that during the subsequent increasing of the transverse pressure plastic deformations take place for $\rho \in S_{p0}$, $\rho \in [\alpha, \eta]$ where η is a previously

unknown constant. The plastic region corresponds to the yield regime DA (Fig. 3.2). Thus, for $\rho \in (\alpha, \eta)$

$$m_2 = -\gamma_0. \quad (3.24)$$

The distribution of the radial bending moment m_1 can be calculated by (3.13) taking $j = 0$. As at $\rho = \beta$, $m_1 = -\gamma_0$ the arbitrary constant E_0 is to be

$$E_0 = \frac{\alpha p}{2} \left(\frac{\alpha^2}{3} - 1 \right). \quad (3.25)$$

Thus the bending moment is defined as

$$m_1 = -\gamma_0 - \frac{p}{2} \left(\frac{\rho^2}{3} - 1 \right) + \frac{\alpha p}{2\rho} \left(\frac{\alpha^2}{3} - 1 \right) \quad (3.26)$$

for $\rho \in [\alpha, \eta]$.

The transverse deflection has the form (3.14) in the plastic region S_{p0} . According to the boundary condition $w(\alpha) = 0$ must be $B_0 = -A_0\alpha$. Thus

$$w = A_0(\rho - \alpha) \quad (3.27)$$

for $\rho \in [\alpha, \eta]$.

Since we can now define $w(\eta)$, $w'(\eta)$, $m_1(\eta)$ the subsequent solution procedure is similar to that accomplished in the previous section. Note that for $\rho > \eta$ the plate is elastic. For determination of the parameter η one has to use the continuity of the moment m_2 taking $m_2(\eta) = -\gamma_0$.

3.7 Several plastic regions

The previous stage of loading terminates at the moment when plastic yielding takes place in the section S_{p1} or in another section. Let us assume for the conceteness sake that the stress profile reaches to the corresponding yield level at $\rho = \alpha_1$ when $p = p_2$. Thus for $p \geq p_2$ the plastic deformations take place in the region (α_1, η_1) as well as in (α, η) which continues the extension. It means that $m_2 = -\gamma_0$ for $\rho \in (\alpha_1, \eta_1)$.

The bending moment distribution for $\rho \in S_{p1}$ can be defined according to (3.13) with the unknown constant E_1 . Similarly the deflection w is given by (3.14) with unknown constants A_1 , B_1 .

The procedure of the determination of constants of integration is similar to that accomplished in the previous case. Now we have to take into account that the region (η, α_1) between plastic regions remain elastic. Here we have unknown constants A_{21} , A_{31} , A_{41} . The number of unknowns in each region is, thus, three. For determination of these constants the continuity requirements for m_1 , w , w' are applicable.

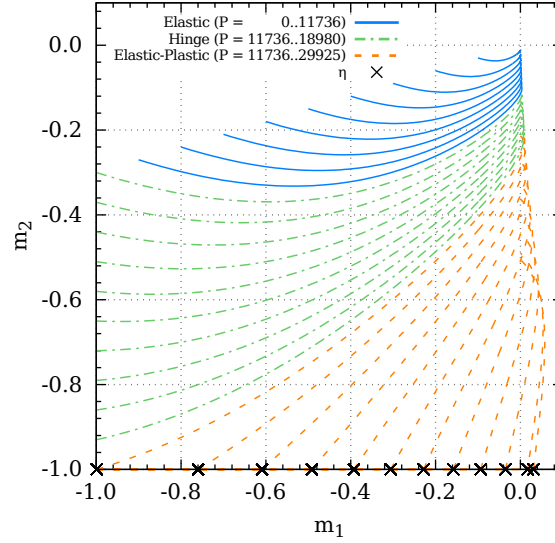


Figure 3.3: Stress profile.

Finally, the parameters η , η_1 are to be determined from equations $m_2(\eta) = -\gamma_0$ and $m_2(\eta_1) = -\gamma_1$ where m_2 is calculated for an elastic region according to (3.20) with previously defined constants A_{2j} , A_{3j} .

3.8 Numerical results

Results of calculations in the case of plates with a single step are presented in Fig. 3.3 – 3.6. The results regard to the plate with inner radius $a = 0.2R$.

The stress profiles on the plane of moments m_1 , m_2 are shown in Fig. 3.3 for different values on the load intensity. It can be seen from Fig. 3.3 that the profiles corresponding to smaller values of the load p lie wholly inside the square $|m_1| \leq 1$, $|m_2| \leq 1$. When the load intensity increases until $p = p_1$ the end of the profile reaches the side $m_1 = -1$ and for $p = p_2$ the corner point where $m_1 = m_2 = -1$. During subsequent growth of the load intensity the end of the stress profile lies on the side $m_2 = -1$ as it was expected theoretically.

Distributions of bending moments m_1 and m_2 are presented in Fig. 3.4 and Fig. 3.5, respectively. The locations of boundaries between elastic and plastic regions in Fig. 3.4 – 3.6 are shown by asterisks. It can be seen from Fig. 3.5 that when the load increases the stress state tends to the pure plastic state. In the case of a plate

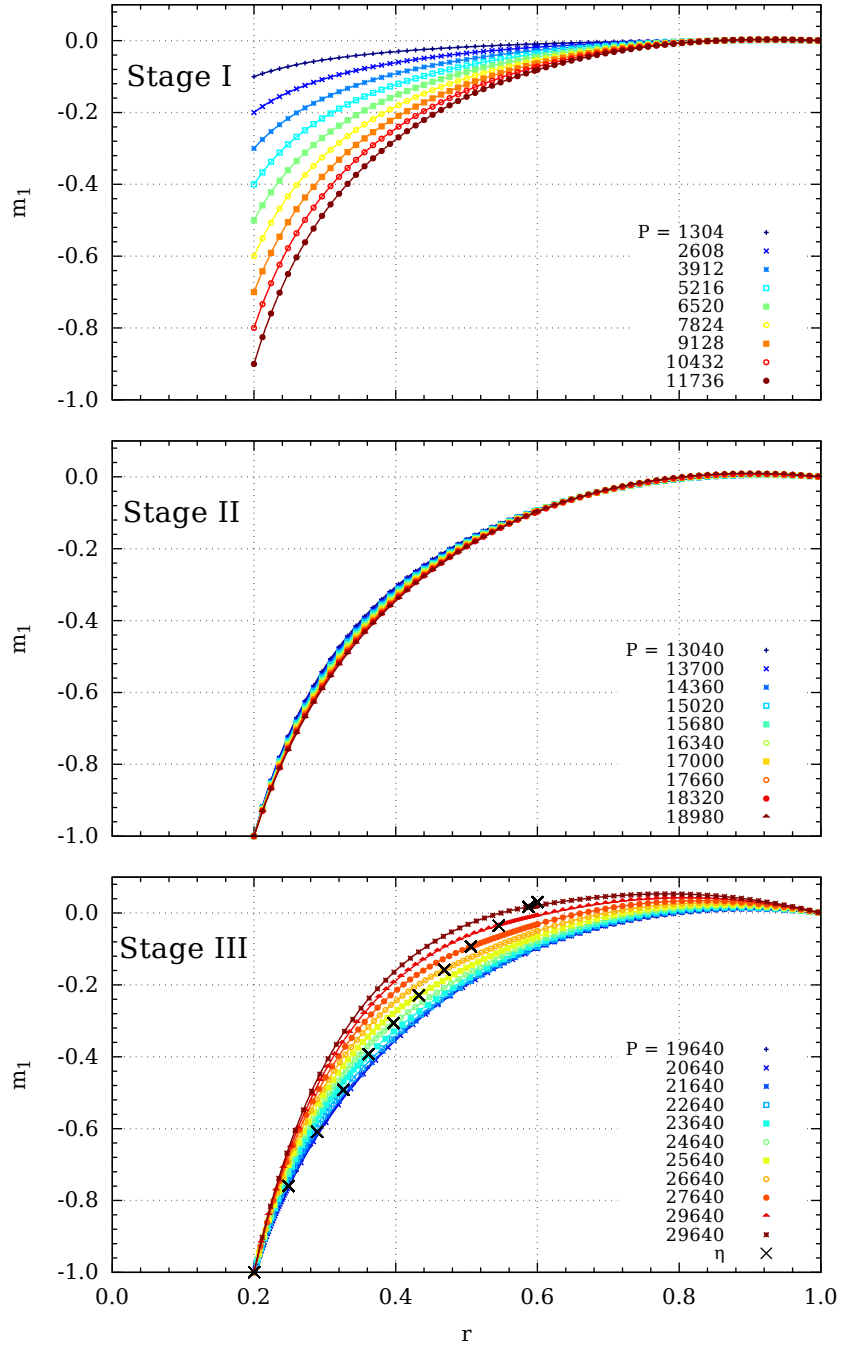


Figure 3.4: Radial bending moments m_1 .

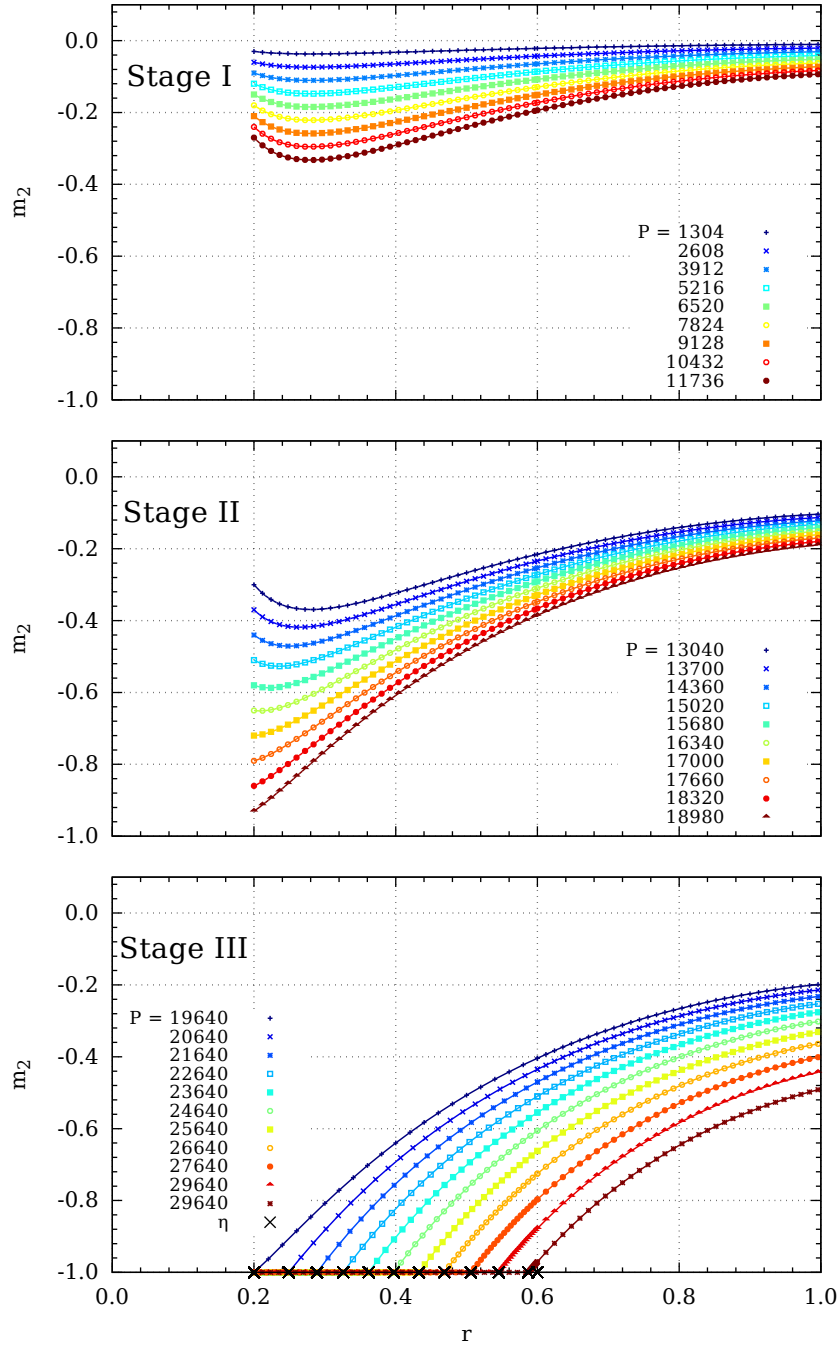


Figure 3.5: Circumferential moments m_2 .

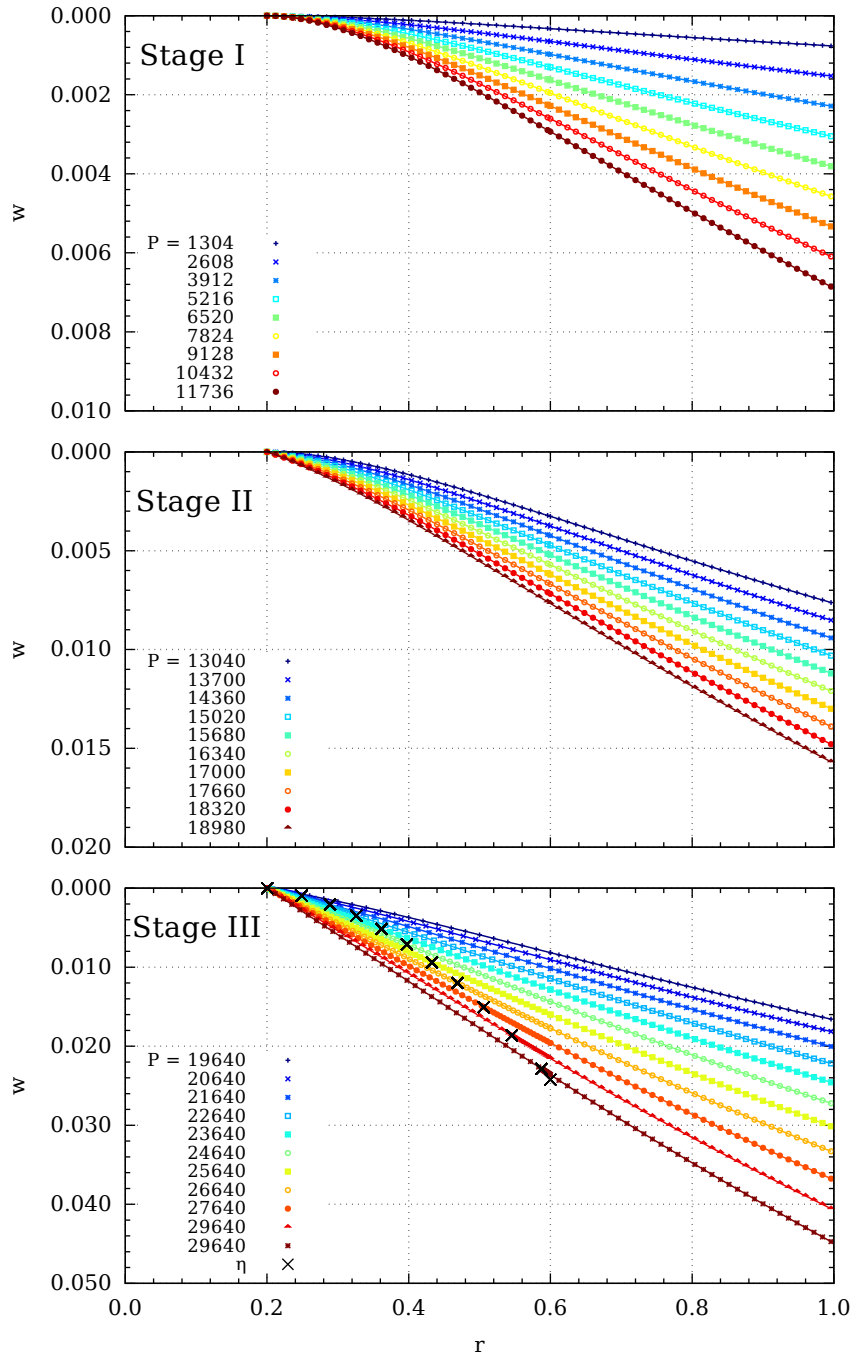


Figure 3.6: Deflections w .

of constant thickness in the pure plastic state $m_2 \equiv -1$. In the case of a stepped plate it can be such that $m_2 = -\gamma_j$ for $\rho \in (\alpha_j, \alpha_{j+1})$, $j = 0, \dots, n$. However, the question which is the stress state at the plastic collapse can be answered by the limit analysis of the plate of particular shape.

3.9 Concluding remarks

A method for theoretical investigation of axisymmetric plates subjected to the distributed transverse pressure was developed. The material of plates was assumed to be an ideal elastic plastic material obeying the square yield condition and the associated flow law in the range of inelastic deformations. In order to get maximal simplicity of the posed problem hardening of the material as well as geometrical non-linearity of the plate behaviour were neglected.

It was assumed that the plates under consideration had piecewise constant thickness with arbitrary number of steps. Exact solutions were developed for the case when the plate is clamped at the inner edge whereas the outer edge is absolutely free. As a result of the solution procedure a succession of stress states which are in equilibrium with the external loading were constructed that led from the wholly elastic to the elastic plastic state and finally to the plastic collapse state.

CHAPTER 4

OPTIMIZATION OF ELASTIC CIRCULAR PLATES WITH ADDITIONAL SUPPORTS

4.1 Optimality criteria

The plate under consideration is simply supported at the edge and it is resting on an absolutely rigid ring support of unknown radius $r = s$. From practical considerations it is evident that the desirable position of the additional support is such that the maximal deflection of the plate is as small as possible. Thus the optimal location of the internal support should minimize the functional

$$J_1 = \max_{r \in [0, R)} W(r, P, s) \quad (4.1)$$

for given loading $P = P(r)$ and thickness $h = h(r)$. However, the cost function presented in the form (4.1) has several drawbacks. First of all, it is a non-differentiable and non-additive functional. The use of non-differentiable functionals in the solution of problems of optimization is quite complicated. On the other hand, the functional (4.1) ignores the expenditures necessary for manufacturing of the additional support.

It can be shown that an approximation of the functional (4.1) can be presented as [4, 47]

$$J_2 = \left(\int_0^R W^k r dr \right)^{\frac{1}{k}} \quad (4.2)$$

where k is an integer. If $k \rightarrow \infty$ then $J_2 \rightarrow \|W\|$.

Due to the circumstances mentioned above in the present chapter the cost function

$$J = \int_0^R W^k r dr + 2\mu_0\pi s \quad (4.3)$$

will be employed. In (4.3) μ_0 stands for the specific cost (cost per unique length) of the additional support. We assume herein that the material cost of the additional support is proportional to its length.

The aim of the chapter is to determine the design of the plate with an additional support (see Fig. 4.1) which minimizes the cost function (4.3) so that at each value of P governing equations of the theory of thin axisymmetric plates with appropriate boundary conditions are satisfied.

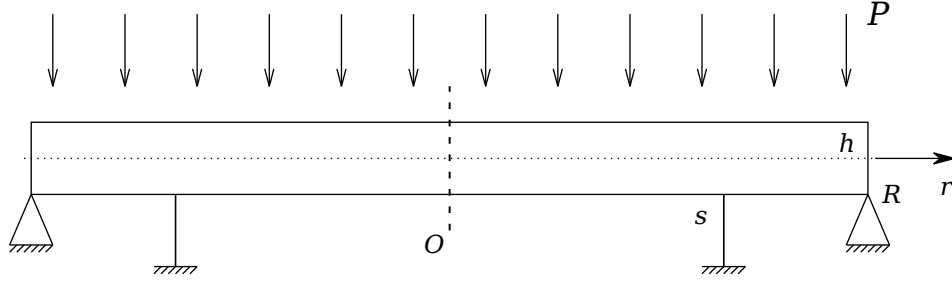


Figure 4.1: Circular plate with additional support.

4.2 Boundary conditions

Taking a look at the equilibrium and constitutive equations (1.1), (1.4), (1.8), (1.11) it appears that one can eliminate from the set of basic equations variables $\sigma_1, \varepsilon_1, \sigma_2, \varepsilon_2, \varkappa_1, \varkappa_2$ and also M_2 . Introducing another new variable Z one can present the system of governing equations as

$$\begin{aligned} \frac{dW}{dr} &= Z, \\ \frac{dZ}{dr} &= -\frac{M_1}{D} - \frac{\nu Z}{r}, \\ \frac{dM_1}{dr} &= \frac{D(\nu^2 - 1)Z}{r^2} - \frac{M_1(1 - \nu)}{r} + Q, \\ \frac{dQ}{dr} &= -\frac{Q}{r} - P(r), \end{aligned} \tag{4.4}$$

where the flexural stiffness D is presented by formula 1.12.

Variables W, Z, M_1, Q will be treated as state variables which satisfy the state equations (4.4) with appropriate boundary and intermediate conditions. At the outer edge of the plate, e. g. at $r = R$ bending moment M_1 and the deflection W must vanish. Thus

$$M_1(R) = 0, \quad W(R) = 0. \tag{4.5}$$

Due to the symmetry at the center of the plate

$$\frac{dW}{dr}(0) = 0, \quad Q(0) = 0. \tag{4.6}$$

At $r = s$ where the rigid ring support is located must be

$$W(s) = 0. \tag{4.7}$$

Note that state variables W , Z , M_1 are continuous whereas Q can be discontinuous at $r = s$.

4.3 Necessary optimality conditions

In order to establish the requirements to be satisfied by the optimal solution let us introduce the augmented functional (see Bryson [10], Hull [30]; Lellep, Polikarpus [49])

$$J_* = \mu s + \int_0^s F_* dr + \int_s^R F_* dr \quad (4.8)$$

where according to (4.3), (4.4)

$$\begin{aligned} F_* = & W^k + \psi_1 \left(\frac{dW}{dr} - Z \right) + \psi_2 \left(\frac{dZ}{dr} + \frac{M_1}{D} + \frac{\nu Z}{r} \right) \\ & + \psi_3 \left(\frac{dM_1}{dr} - \frac{D(\nu^2 - 1)Z}{r^2} + \frac{M_1(1 - \nu)}{r} - Q \right) \\ & + \psi_4 \left(\frac{dQ}{dr} + \frac{Q}{r} + P(r) \right) \end{aligned} \quad (4.9)$$

and $\mu = 2\pi\mu_0$, the quantities $\psi_1 - \psi_4$ being adjoint variables.

Evidently the problem posed above belongs to the class of optimal control problems with moving boundaries. Therefore, one has to employ total variations when deriving necessary conditions of minimum of the functional (4.8). The total variation of a state variable y at $r = s + 0$ or at $r = s - 0$ must be calculated by the following sample:

$$\Delta y(s \pm 0) = \delta y(s \pm 0) + \frac{dy(s \pm 0)}{dr} \cdot \Delta s \quad (4.10)$$

where Δy is the total variation and δy stands for the ordinary variation of the variable y . If the state variable is continuous at $r = s$ then $\Delta y(s - 0) = \Delta y(s + 0) = \Delta y(s)$. However, in the case of discontinuous variables one has to distinguish the quantities $\Delta y(s - 0)$ and $\Delta y(s + 0)$. Note that even in the case of continuous variables the quantities $\delta y(s - 0)$ and $\delta y(s + 0)$ must not be equal to each other.

The total variation of a Lagrange' functional is calculated by the rule (see Bryson [10], Lellep [46]),

$$\Delta \int_0^s F dr = \delta \int_0^s F dr + F|_s \cdot \Delta s \quad (4.11)$$

where Δs stands for an arbitrary increment of s . According to (4.11) one can write

$$\Delta J_* = \mu \Delta s + \delta \int_0^s F_* dr + \delta \int_s^R F_* dr + F_*|_{s-} \Delta s - F_*|_{s+} \Delta s. \quad (4.12)$$

Taking (4.9) into account one can easily determine the following weak variation

$$\begin{aligned}
\delta \int_a^b F_* dr = \int_a^b & \left\{ kW^{k-1} r \delta W - \frac{d\psi_1}{dr} \delta W - \psi_1 \delta Z - \frac{d\psi_2}{dr} \delta Z \right. \\
& + \frac{\psi_2}{D} \delta M_1 + \frac{\nu \psi_2}{r} \delta Z - \frac{d\psi_3}{dr} \delta M_1 - \frac{D(\nu^2 - 1) \psi_3}{r^2} \delta Z \\
& + \frac{\psi_3(1 - \nu)}{r} \delta M_1 - \psi_3 \delta Q - \frac{d\psi_4}{dr} \delta Q + \frac{\psi_4}{r} \delta Q \left. \right\} dr \\
& + (\psi_1 \delta W + \psi_2 \delta Z + \psi_3 \delta M_1 + \psi_4 \delta Q) \Big|_a^b
\end{aligned} \tag{4.13}$$

where a and b are arbitrary boundaries of integration.

Substituting the both integrals in (4.12) by (4.13) with appropriate choice of boundaries a and b leads to the relation

$$\begin{aligned}
\Delta J_* = \mu \Delta s + \int_0^R & \left\{ kW^{k-1} r \delta W - \frac{d\psi_1}{dr} \delta W - \psi_1 \delta Z - \frac{d\psi_2}{dr} \delta Z \right. \\
& + \frac{\psi_2}{D} \delta M_1 + \frac{\nu \psi_2}{r} \delta Z - \frac{d\psi_3}{dr} \delta M_1 - \frac{D(\nu^2 - 1) \psi_3}{r^2} \delta Z \\
& + \frac{\psi_3(1 - \nu)}{r} \delta M_1 - \psi_3 \delta Q - \frac{d\psi_4}{dr} \delta Q + \frac{\psi_4}{r} \delta Q \left. \right\} dr \\
& + (\psi_1 \delta W + \psi_2 \delta Z + \psi_3 \delta M_1 + \psi_4 \delta Q) \Big|_0^s \\
& + (\psi_1 \delta W + \psi_2 \delta Z + \psi_3 \delta M_1 + \psi_4 \delta Q) \Big|_s^R
\end{aligned} \tag{4.14}$$

where the matter that $F_*(s) = 0$ has taken into account.

Making use of (4.14) one easily obtains from the equation $\Delta J_* = 0$ the system of adjoint equations

$$\begin{aligned}
\frac{d\psi_1}{dr} &= rkW^{k-1}, \\
\frac{d\psi_2}{dr} &= -\psi_1 + \frac{\nu \psi_2}{r} - \frac{D(\nu^2 - 1) \psi_3}{r^2}, \\
\frac{d\psi_3}{dr} &= \frac{\psi_2}{D} + \frac{\psi_3(1 - \nu)}{r}, \\
\frac{d\psi_4}{dr} &= -\psi_3 + \frac{\psi_4}{r}.
\end{aligned} \tag{4.15}$$

Note that although the adjoint set (4.15) holds good for each $r \in [0, R]$ it must be integrated separately in regions $(0, s)$ and (s, R) , respectively. The reason is that some of adjoint variables can be discontinuous at $r = s$.

Boundary conditions (4.5), (4.6) admit to present the transversality conditions as

$$\psi_1(0) = 0, \quad \psi_3(0) = 0 \quad (4.16)$$

and

$$\psi_2(R) = 0, \quad \psi_4(R) = 0. \quad (4.17)$$

Substituting (4.15) – (4.17) in (4.14) admits to rewrite the equation $\Delta J_* = 0$ as

$$\mu \Delta s - (\psi_1 \delta W + \psi_2 \delta Z + \psi_3 \delta M_1 + \psi_4 \delta Q)|_{s-0}^{s+0} = 0. \quad (4.18)$$

From the physical considerations it is evident that W , Z and M_1 are continuous at $r = s$. Thus following (4.10) one can write

$$\begin{aligned} \delta W(s \pm 0) &= \Delta W(s) - \frac{dW}{dr}(s) \cdot \Delta s, \\ \delta Z(s \pm 0) &= \Delta Z(s) - \frac{dZ}{dr}(s) \cdot \Delta s, \\ \delta M_1(s \pm 0) &= \Delta M_1(s) - \frac{dM_1(s \pm 0)}{dr} \cdot \Delta s, \\ \delta Q(s \pm 0) &= \Delta Q(s \pm 0) - \frac{dQ(s \pm 0)}{dr} \cdot \Delta s. \end{aligned} \quad (4.19)$$

Substituting the weak variations of state variables from (4.19) to (4.18) and taking into account that $\Delta W(s) = 0$ and $\Delta Z(s)$, $\Delta M_1(s)$, $\Delta Q(s \pm 0)$ are independent leads to the requirements

$$\begin{aligned} \psi_2(s - 0) - \psi_2(s + 0) &= 0, \\ \psi_3(s - 0) - \psi_3(s + 0) &= 0 \end{aligned} \quad (4.20)$$

and

$$\psi_4(s - 0) = \psi_4(s + 0) = 0. \quad (4.21)$$

It was assumed above that Z and M_1 are continuous everywhere; thus in particular at $r = s$. Bearing in mind the continuity of M_1 it infers from (1.8) and (1.11) that $\kappa_1 = -\frac{dZ}{dr}$ is also continuous at $r = s$.

Substituting (4.19) – (4.21) in (4.18) and taking into account the continuity of Z , κ_1 , κ_2 and ψ_2 , ψ_3 , also the arbitrariness of the increment Δs one can present (4.18) as

$$\mu + [\psi_1(s)] \frac{dW(s)}{dr} + \psi_3(s) \left[\frac{dM_1(s)}{dr} \right] = 0. \quad (4.22)$$

In (4.22) the quadratic brackets denote the finite jumps of corresponding variables at $r = s$, e. g.

$$[y(s)] = y(s + 0) - y(s - 0)$$

where $y(s \pm 0)$ stands for right and left hand limits of the discontinuous variable $y(r)$ at $r = s$.

4.4 Solution of governing equations

Consider the solution of state equations (4.4) in greater detail in the case when the plate thickness h is constant. In this case it follows from (1.12) that $D = \text{const}$, as well.

Integrating the last equation in the system (4.4) one obtains

$$Q = -\frac{1}{r} \left(\int P(r) dr + C_{\pm} \right) \quad (4.23)$$

where C_+ and C_- stand for constants of integration in the regions $[0, s]$ and $[s, R]$, respectively. According to formula (1.15) the general solution can be presented as

$$W = \frac{Pr^4}{64D} + A_{1j}r^2 \ln r + A_{2j}r^2 + A_{3j} \ln r + A_{4j} \quad (4.24)$$

for $r \in [r_j, r_{j+1}]$ and $j = 0, 1$. Here the following notation is used: $r_0 = 0$, $r_1 = s$ and $r_2 = R$. Evidently,

$$Z = \frac{Pr^3}{16D} + A_{1j}r(2 \ln r + 1) + 2A_{2j}r + \frac{A_{3j}}{r} \quad (4.25)$$

and

$$\begin{aligned} M_1 &= -\frac{Pr^2(3 + \nu)}{16} - A_{1j}D[3 + \nu + 2(1 + \nu) \ln r] \\ &\quad - 2DA_{2j}(1 + \nu) - \frac{D(\nu - 1)}{r^2}A_{3j}, \\ M_2 &= -\frac{Pr^2(1 + 3\nu)}{16} - A_{1j}D[1 + 3\nu + 2(1 + \nu) \ln r] \\ &\quad - 2DA_{2j}(1 + \nu) - \frac{D(\nu - 1)}{r^2}A_{3j}. \end{aligned} \quad (4.26)$$

The integration constants $A_{1j} - A_{4j}$ will be determined from the boundary and continuity conditions. Let us consider first the solution in the internal region for $r \in [0, s]$. Here $j = 0$ in (4.24) – (4.26). Since at the center of the plate the quantities $W(0)$, $M_1(0)$, $M_2(0)$ must be finite whereas according to (4.6) $Z(0) = 0$ one has

$$A_{10} = 0, \quad A_{30} = 0. \quad (4.27)$$

Boundary conditions (4.5) with (4.7) and the continuity requirements for Z and M_1 result in

$$\begin{aligned}
A_{20}s^2 + A_{40} + \frac{Ps^4}{64D} &= 0, \\
A_{11}s^2 \ln s + A_{21}s^2 + A_{31} \ln s + A_{41} + \frac{Ps^4}{64D} &= 0, \\
-2sA_{20} + A_{11}s(1 + 2 \ln s) + 2A_{21}s - \frac{A_{31}}{s} &= 0, \\
-2A_{20} + A_{11}(3 + 2 \ln s) + 2A_{21} - \frac{A_{31}}{s^2} &= 0, \\
A_{11}R^2 \ln R + A_{21}R^2 + A_{31} \ln R + A_{41} + \frac{PR^4}{64D} &= 0, \\
A_{11}[2(1 + \nu) \ln R + 3 + \nu] + 2(1 + \nu)A_{21} \\
- \frac{A_{31}(1 - \nu)}{R^2} + \frac{PR^2(3 + \nu)}{16D} &= 0.
\end{aligned} \tag{4.28}$$

The system (4.28) can be easily solved with respect to unknowns A_{20} , A_{40} , A_{11} , A_{21} ,

A_{31} , A_{41} and presented as

$$\begin{aligned}
A_{20} &= \frac{p}{K} \cdot [R^6[2(\nu+5)(\ln s - \ln R) + 3\nu + 13] \\
&+ s^2 R^4[4(\nu+3)(\ln s - \ln R) - 3\nu - 13] \\
&+ s^4 R^2[2(\nu+1)(\ln s - \ln R) + \nu - 1] + s^6(1 - \nu)] , \\
A_{40} &= \frac{-ps^2 R^2}{K} \cdot [R^4[2(\nu+5)(\ln s - \ln R) + 3\nu + 13] \\
&+ 4s^2 R^2[(\nu+3)(\ln s - \ln R) - \nu - 4] \\
&+ s^4[-2(\nu+1)(\ln s - \ln R) + \nu + 3]] , \\
A_{11} &= \frac{2pR^2}{K} \cdot [(\nu+5)R^4 - 2(\nu+3)s^2 R^2 + (\nu+1)s^4] , \\
A_{21} &= \frac{-p}{K} [R^6[2(\nu+5) \ln R - \nu - 3] \\
&+ s^2 R^4[4(\nu+3)(\ln R - 2 \ln s) - \nu + 1] \\
&+ s^4 R^2[2(\nu+1) \ln R + \nu + 3] + s^6(\nu - 1)] , \\
A_{31} &= s^2 \cdot A_{11}, \\
A_{41} &= \frac{-ps^2 R^2}{K} [R^4[2(\nu+5)(2 \ln s - \ln R) + \nu + 3] \\
&- 4s^2 R^2[(\nu+3) \ln R + 1] + s^4[2(\nu+1) \ln R - \nu + 1]] ,
\end{aligned} \tag{4.29}$$

where $K = 64D [(\nu-1)s^4 - (\nu+3)R^4 + 4s^2 R^2[(\nu+1)(\ln R - \ln s) + 1]]$.

4.5 Solution of the adjoint system

The adjoint system (4.15) can be integrated after the substitution of (4.24) in (4.15). For the sake of simplicity let us consider the case when $k = 1$ in greater detail.

It is easy to recheck that the general solution of (4.15) corresponding to the case

$k = 1$ can be presented as

$$\begin{aligned}
\psi_1 &= \frac{r^2}{2} + C_{1j}, \\
\psi_2 &= C_{2j}r + \frac{C_{3j}}{r} - \frac{(3+\nu)r^3}{16} - \frac{C_{1j}(1+\nu)r \ln r}{2}, \\
\psi_3 &= \frac{C_{2j}r^2}{D(\nu+1)} + \frac{C_{3j}}{D(\nu-1)} - \frac{r^4}{16D} \\
&\quad - \frac{C_{1j}r^2}{D(\nu^2-1)} + \frac{C_{1j}r^2[(1-\nu) \ln r + 1]}{2D(\nu-1)}, \\
\psi_4 &= -\frac{C_{2j}r^3}{2D(\nu+1)} - \frac{C_{3j}r \ln r}{D(\nu-1)} + C_{4j}r \\
&\quad + \frac{r^5}{64D} + \frac{C_{1j}r^3 \ln r}{4D} + \frac{C_{1j}(3-2\nu-\nu^2)r^3}{8D(\nu^2-1)}
\end{aligned} \tag{4.30}$$

for $r \in [r_j, r_{j+1}]$ where $j = 0, 1$.

For determination of 8 unknown constants $C_{1j}, C_{2j}, C_{3j}, C_{4j}$ where $j = 0, 1$ one has 8 boundary and intermediate conditions presented by (4.16), (4.17), (4.20) and (4.21).

It immediately follows from boundary conditions (4.16) that

$$C_{10} = 0, \quad C_{30} = 0. \tag{4.31}$$

The boundary and intermediate conditions (4.17), (4.20), (4.21) lead to the linear

algebraic system

$$\begin{aligned}
& C_{21}R + \frac{C_{31}}{R} - \frac{C_{11}(1-\nu)R \ln R}{2} - \frac{(3-\nu)R^3}{16} = 0, \\
& -\frac{C_{21}R^3(\nu-1)}{2} - C_{31}(\nu+1)R \ln R + C_{41}(\nu^2-1)RD \\
& + \frac{R^5(\nu^2-1)}{64} + \frac{C_{11}(\nu^2-1)R^3 \ln R}{4} + \frac{C_{11}(3-2\nu-\nu^2)R^3}{8} = 0, \\
& -\frac{C_{21}s^3(\nu-1)}{2} - C_{31}(\nu+1)s \ln s + C_{41}(\nu^2-1)sD \\
& + \frac{s^5(\nu^2-1)}{64} + \frac{C_{11}(\nu^2-1)s^3 \ln s}{4} + \frac{C_{11}(3-2\nu-\nu^2)s^3}{8} = 0, \quad (4.32) \\
& -\frac{C_{20}s^3(\nu-1)}{2} + C_{40}s(\nu^2-1)D + \frac{s^5(\nu^2-1)}{64} = 0, \\
& (C_{21} - C_{20})s + \frac{C_{31}}{s} - \frac{C_{11}(1+\nu)s \ln s}{2} = 0, \\
& (C_{21} - C_{20})(\nu-1)s^2 + C_{31}(\nu+1) - C_{11}s^2 \\
& + \frac{C_{11}s^2(\nu+1)[(1-\nu) \ln s + 1]}{2} = 0.
\end{aligned}$$

From (4.32) one can easily determine the unknown constants C_{20} , C_{40} , C_{11} , C_{21} , C_{31} , C_{41} and present as

$$\begin{aligned}
C_{20} &= C_{21} + \frac{C_{31}}{s^2} - \frac{C_{11}(1+\nu) \ln s}{2}, \\
C_{40} &= \frac{C_{20}s^2}{2D(\nu+1)} - \frac{s^4}{64D}, \\
C_{11} &= \frac{R^2(R^2 - s^2)[2R^2(3+\nu) - (\nu+1)(R^2 + s^2)]}{8L}, \\
C_{21} &= \frac{C_{11}s^2(\nu-1)}{4R^2} + \frac{(3+\nu)R^2}{16} + \frac{C_{11}(1+\nu) \ln R}{2}, \quad (4.33) \\
C_{31} &= \frac{-C_{11}s^2(\nu-1)}{4}, \\
C_{41} &= \frac{C_{21}s^2}{2D(\nu+1)} + \frac{C_{31} \ln s}{D(\nu-1)} - \frac{s^4 + 16C_{11}s^2 \ln s}{64D} \\
& - \frac{C_{11}(-\nu^2 - 2\nu + 3)s^2}{8D(\nu^2 - 1)},
\end{aligned}$$

where

$$L = s^2(\nu - 1)(s^2 - R^2) + 2R^2(\nu + 1)(s^2 - R^2) \ln R - 2R^2 s^2(\nu + 1) \\ (\ln s - \ln R) + 2R^2(\nu + 1)(R^2 \ln R - s^2 \ln s) - R^2(\nu + 3)(R^2 - s^2).$$

4.6 Discussion of results

Results of calculations are presented in Fig. 4.2 – 4.4. The calculations are implemented for $k = 1$ and $\mu = 0$.

In Fig. 4.2 the distributions of deflections of the plate are presented for various values of the transverse load intensity. The first and the third graph in Fig. 4.2 and correspond to the positions of the support at $s = 0.2R$ and $s = 0.7R$, respectively whereas the second one is associated with the optimal location of the intermediate support. The optimal solution corresponds to $s = 0.526R$. It can be seen from Fig. 4.2 that in the case of $s = 0.2R$ deflections at the central part of the plate for $r < 0.2R$ are directed upward despite the pressure is directed downward. Similarly in the case when $s = 0.7R$ one can see negative deflections in the outward region for $r > 0.7R$. However, in the case of optimal position of the additional support the deflections are non-negative everywhere. It is somewhat surprising that the maximal deflections in the central and outward regions of the plate, respectively, are quite different in the optimal case. However, one has to take into account that the cost function (4.3) with $\mu_0 = 0$, $k = 1$ corresponds to the volume of the axisymmetric body.

In Fig. 4.3 bending moment M_1 is presented for the cases when $s = 0.2R$, $s = 0.526R$ (optimal case) and $s = 0.7R$. It can be seen from Fig. 4.3 that the slope of the radial bending moment has finite jumps at the support position, as might be expected. It is some what surprising that the radial bending moment vanishes at an internal point for any values of the transverse pressure loading. It reveals from Fig. 4.3 that in the case of smaller values of the radius of the internal support the radial bending moment remains negative in the central part of the plate. It is negative in the vicinity of the support in the optimal case, as well.

Distributions of the circumferential bending moment M_2 are presented in Fig. 4.4 for different values of the pressure loading. It reveals from Fig. 4.4 that the bending moment M_2 is unexpectedly continuous at $r = s$.

The efficiency of the design established can be assessed by the ratios

$$e_{1,2} = \frac{J_0}{J_{1,2}}$$

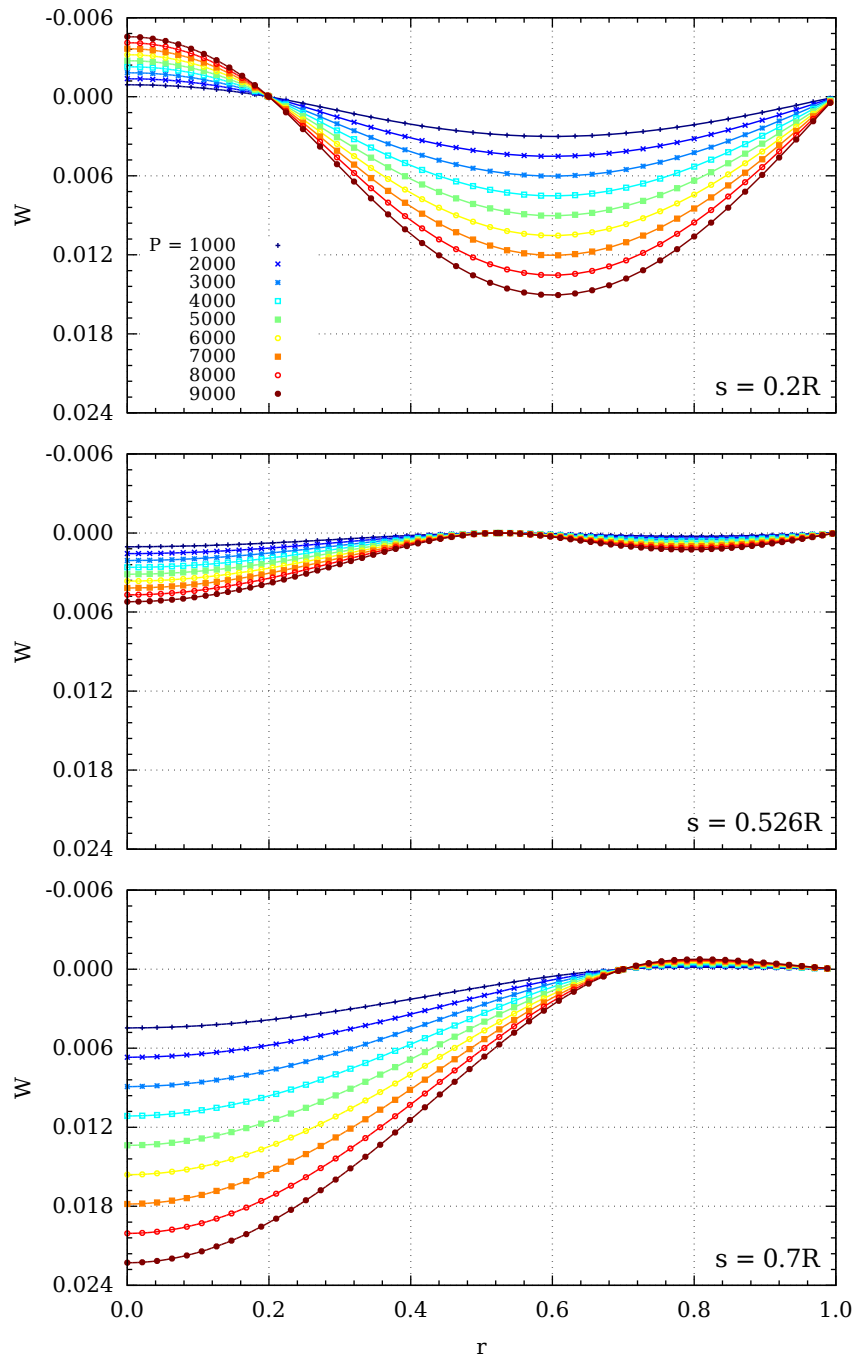


Figure 4.2: Deflections.

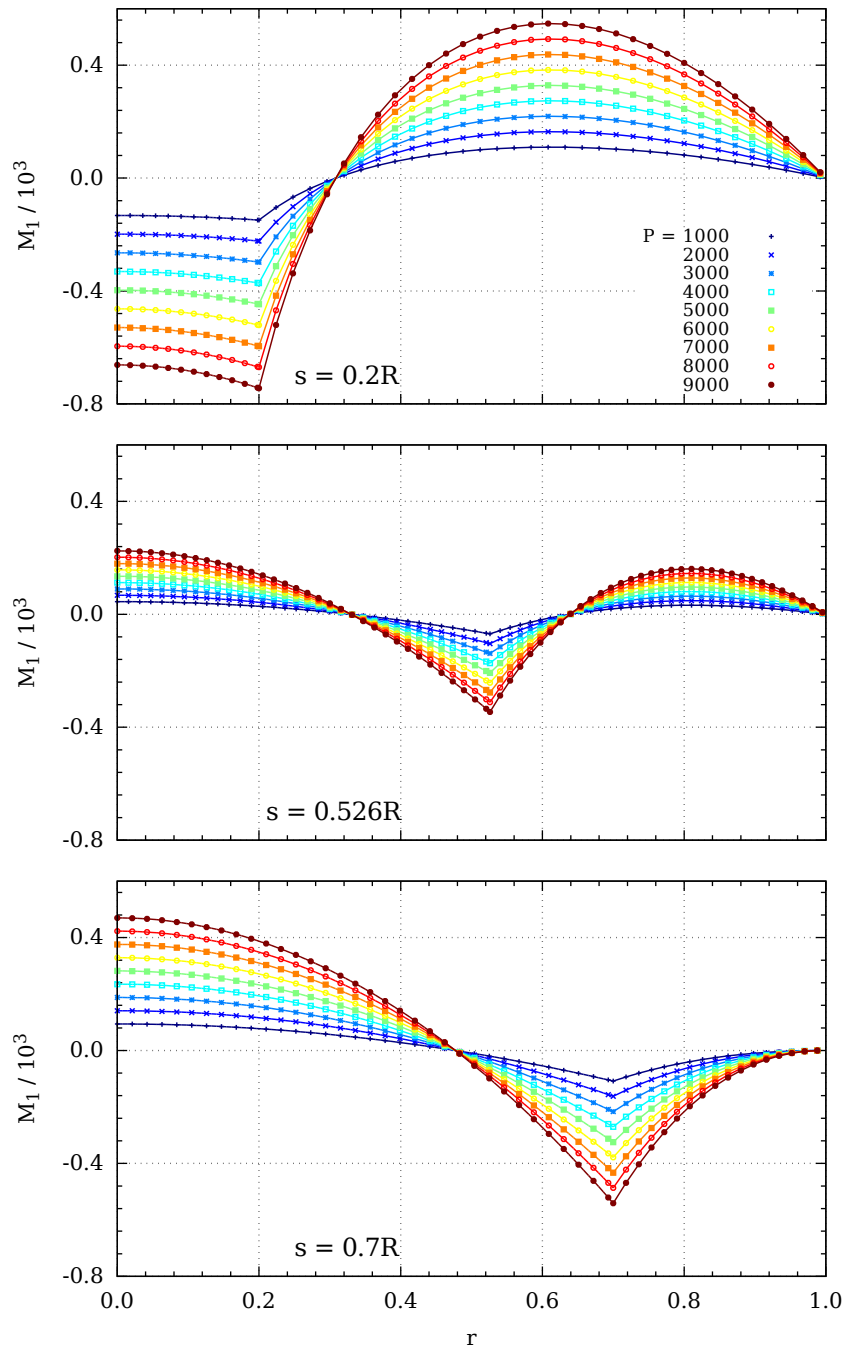


Figure 4.3: Radial bending moments.

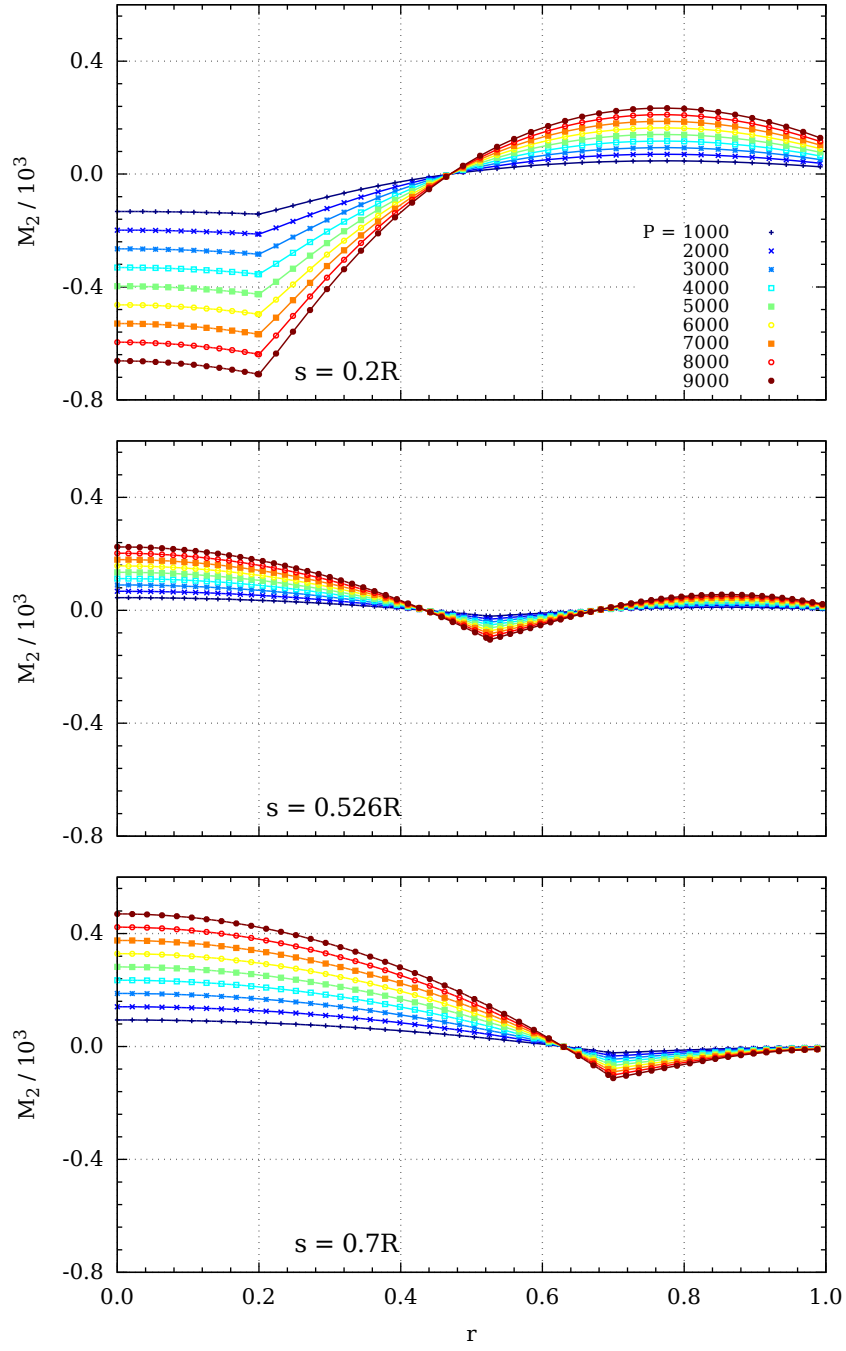


Figure 4.4: Circumferential moments.

Table 4.1: Efficiency of the design.

s/R	0.2	0.4	0.526	0.6	0.8
$10^{-4}J_0/R^6$	1.996	1.055	1.027	1.031	1.849
10^2e_1	1.562	0.826	0.803	0.807	1.447
10^2e_2	43.80	23.15	22.54	22.62	40.58

where J_0 stands for the value of the cost function (4.3) corresponding to the optimal position of the internal support. However,

$$J_1 = \int_0^R W r dr$$

in the case of the plate without additional supports and J_2 stands for the value of J in the case when internal support is located at the center of the plate. Calculations carried out showed that the value of the cost function is very sensitive with respect to the location of the internal support (Table 4.1). It can be seen from Table 4.1 that in the case of the optimal location of the internal support the value of J is more than four times less than that corresponding to the plate with a support at its center.

4.7 Concluding remarks

Variational methods of the theory of optimal control are used for solving the problem of optimal location of an additional rigid ring support in the case of a circular plate. The plate is made of an elastic material and subjected to a distributed transverse pressure. Necessary optimality conditions have derived under the assumption that the cost of the additional support is proportional to its length. Numerical results have presented for the plate simply supported at the edge and subjected to the uniformly distributed transverse pressure.

The results of calculations showed that the optimal position of the additional support admits to diminish essentially the cost function. It revealed by calculations that the both, radial and circumferential bending moments are continuous over the entire plate.

CHAPTER 5

ANISOTROPIC PLATE

5.1 The optimization problem

Let us consider a circular plate of radius R made of a unidirectionally reinforced composite material. It is assumed that the fibers are embedded in the matrix material more or less uniformly with enough high density so that the material can be modelled as a quasi homogeneous anisotropic material. In the present chapter two cases of unidirectional reinforcement will be considered. These are the radial and the circumferential reinforcements, respectively.

The plate under consideration is simply supported at the edge and it is subjected to the axisymmetric transverse pressure loading of intensity $P(r)$ where r stands for the current radius. Under these assumptions it is reasonable to presume that the stress strain state of the plate remains axisymmetric during the deformation.

The thickness h of the plate is assumed to be piecewise constant, e.g. $h = h_j$ for $r \in (a_j, a_{j+1})$ where $j = 0, \dots, n$. Note that in the case of a sandwich plate the quantity h is interpreted as the thickness of carrying layers.

The parameters h_j, a_j are treated as preliminarily unknown parameters to be determined so that maximal deflection $W_0 = W(0)$ attains the minimal value under given weight or material volume of the plate. The latter can be presented as

$$V = \pi \sum_{j=0}^n h_j (a_{j+1}^2 - a_j^2). \quad (5.1)$$

It is stipulated in (5.1) that $a_0 = 0, a_{n+1} = R$.

5.2 Equations of equilibrium and strain components

In the present chapter the governing equations (1.1) of the classical bending theory of thin plates will be used.

The generalized stresses are coupled with the principal stresses as

$$\begin{aligned} N_{1,2} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{1,2} dz \\ M_{1,2} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{1,2} z dz \end{aligned} \quad (5.2)$$

where σ_1, σ_2 stand for the normal stresses in the radial and circumferential direction, respectively. Here h and z are the thickness and the coordinate in the transversal direction, respectively.

The principle of virtual work states that

$$\delta E_i = \delta A_e. \quad (5.3)$$

Here δE_i stands for the variation of the internal energy induced by virtual displacements and δA_e is the variation of the work done by external forces. In the case of axisymmetric plates (see Jones [32], Ventsel and Krauthammer [101])

$$\delta E_i = \int_a^R (M_1 \delta \varkappa_1 + M_2 \delta \varkappa_2) r dr \quad (5.4)$$

whereas

$$\delta A_e = \int_a^R P(r) \delta W r dr. \quad (5.5)$$

In (5.5) \varkappa_1 and \varkappa_2 stand for the principal curvatures defined by (1.8).

Note that the principle of virtual work in the form (5.3) – (5.5) holds good in the case of axisymmetric deformations under the condition that shear deformations as well as the in-plane displacements can be disregarded. The in-plane displacements are taken into account in the non-linear plate theories.

5.3 Constitutive equations

It is known that in a unidirectional fiber-reinforced layer of material two perpendicular planes of symmetry exist at each point. The intersections of these planes with the middle plane of the lamina define two directions denoted by subscripts 1 and 2, respectively. These directions called principal axis of orthotropy correspond to the direction of fibers and a direction transverse to the fibers.

In an orthotropic layer the assumptions of the plane stress hold good and thus the Hooke's law reads (see Daniel and Ishai [16], Jones [33])

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [Q] \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \quad (5.6)$$

where $\sigma_1, \sigma_2, \tau_{12}$ are the stress components and $\varepsilon_1, \varepsilon_2, \gamma_{12}$ corresponding strain components. The matrix $[Q]$ is defined as

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}. \quad (5.7)$$

The elements Q_{ij} in (5.7) are so-called reduced stiffnesses related to the material constants. Let us denote by E_1, E_2 Young's moduli in two principal directions and by ν_{12}, ν_{21} corresponding Poisson's ratios.

For any combination of fibers and the matrix (see Jones [33])

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \quad (5.8)$$

and the reduced stiffnesses are

$$\begin{aligned} Q_{11} &= \frac{E_1}{1 - \nu_{12}\nu_{21}}, \\ Q_{22} &= \frac{E_2}{1 - \nu_{12}\nu_{21}}, \\ Q_{12} &= \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}}, \\ Q_{66} &= G_{12}. \end{aligned} \quad (5.9)$$

It can be seen from (5.8), (5.9) that the constitutive relations (5.6), (5.7) can be presented via four independent material constants.

In the following the main attention will be focused at the determination of bending moments assuming that the contribution of membrane forces and in-plane displacements to the stress strain state is relatively small in the range of small displacements. Thus, assuming that the fibers are embedded in the matrix material in the radial or circumferential direction only and that

$$\begin{aligned} \varepsilon_1 &= z\kappa_1, \\ \varepsilon_2 &= z\kappa_2 \end{aligned} \quad (5.10)$$

it follows from (5.6)

$$\begin{aligned} \sigma_1 &= z(Q_{11}\kappa_1 + Q_{12}\kappa_2), \\ \sigma_2 &= z(Q_{12}\kappa_1 + Q_{22}\kappa_2). \end{aligned} \quad (5.11)$$

Substituting (5.11) in (5.2) and taking (5.8), (5.9) into account one obtains

$$\begin{aligned} M_1 &= DE_1(\kappa_1 + \nu_{21}\kappa_2), \\ M_2 &= DE_2(\nu_{12}\kappa_1 + \kappa_2) \end{aligned} \quad (5.12)$$

where

$$D = \frac{h^3}{12(1 - \nu_{12}\nu_{21})}. \quad (5.13)$$

For regions of constant thickness (a_j, a_{j+1}) where $h = h_j = \text{const}$ it is reasonable to denote

$$D_j = \frac{h_j^3}{12(1 - \nu_{12}\nu_{21})}. \quad (5.14)$$

Equations (5.12) with (5.13) present the constitutive equations for anisotropic (in the present case for orthotropic) layers and plates. Substituting the curvatures \varkappa_1, \varkappa_2 by the help of (1.8) in (5.12) one obtains

$$\begin{aligned} M_1 &= -E_1 D_j \left(\frac{d^2 W}{dr^2} + \frac{\nu_{21}}{r} \frac{dW}{dr} \right), \\ M_2 &= -E_2 D_j \left(\frac{\nu_{12} d^2 W}{dr^2} + \frac{1}{r} \frac{dW}{dr} \right) \end{aligned} \quad (5.15)$$

for $r \in (a_j, a_{j+1})$, where $j = 0, \dots, n$.

5.4 Solution of governing equations

Let us consider the case when the intensity of the transverse loading $P(r) = P = \text{const}$ in a greater detail. In this case the third equation in the system (1.1) can be easily integrated. In fact, the differentiation of the second equation in (1.1) and the substitution of the shear force results in

$$\frac{d}{dr} \left[\frac{d}{dr} (rM_1) - M_2 \right] = -Pr. \quad (5.16)$$

The integration of (5.16) gives

$$\frac{d}{dr} (rM_1) - M_2 = -\frac{Pr^2}{2} - B_{0j} \quad (5.17)$$

for $r \in (a_j, a_{j+1})$, $j = 0, \dots, n$. Here B_{0j} stands for an arbitrary constant.

Substituting the bending moments from (5.15) in (5.17) and accounting for the relation (5.8) results in

$$\frac{rd^3 W}{dr^3} + \frac{d^2 W}{dr^2} - \frac{E_2}{E_1} \cdot \frac{1}{r} \frac{dW}{dr} = \frac{Pr^2}{2E_1 D_j} + \frac{B_{0j}}{E_1 D_j} \quad (5.18)$$

for $r \in (a_j, a_{j+1})$. The shear force Q must be continuous at each $r \in [a, R]$. Thus $Q(a_j - 0) = Q(a_j + 0)$. Therefore, one can write

$$B_{0j} = B_0 \quad (5.19)$$

for each $j = 0, \dots, n$. Indeed, it immediately follows from (1.1), (5.17) that

$$Q(r) = \frac{1}{r} \left(-\frac{Pr^2}{2} - B_{0j} \right) \quad (5.20)$$

for $r \in (a_j, a_{j+1})$, $j = 0, \dots, n$. Thus at $r = a_j \pm 0$ one has

$$-\frac{Pa_j}{2} - \frac{B_{0j-1}}{a_j} = -\frac{Pa_j}{2} - \frac{B_{0j}}{a_j}. \quad (5.21)$$

From (5.21) one can see that $B_{0j-1} = B_{0j}$ for each $j = 1, \dots, n$ and therefore, (5.19) holds good.

Let us concentrate on the integration of the equation (5.18). It is reasonable to denote

$$E_2 = k^2 E_1. \quad (5.22)$$

Multiplying the both sides of (5.18) to r and integrating results in the second order differential equation (here C_{0j} is an arbitrary constant)

$$\frac{r^2 d^2 W}{dr^2} - \frac{rdW}{dr} + (1 - k^2)W = \frac{Pr^4}{8E_1 D_j} + \frac{B_0 r^2}{2E_1 D_j} + C_{0j}(1 - k^2) \quad (5.23)$$

for $r \in (a_j, a_{j+1})$. This is a linear non-homogeneous equation. The general solution of (5.23) consists of the sum of the general solution of corresponding homogeneous equation

$$W_h = C_{1j} r^{1+k} + C_{2j} r^{1-k} \quad (5.24)$$

and of a particular solution of (5.23). The latter can be presented as

$$W_p = \frac{Pr^4}{8(9 - k^2)E_1 D_j} + \frac{B_0 r^2}{2(1 - k^2)E_1 D_j} + C_{0j}. \quad (5.25)$$

In (5.24), (5.25) C_{0j} , C_{1j} and C_{2j} are arbitrary constants of integration.

Finally, the general solution of (5.23) for $(r \in a_j, a_{j+1})$, $j = 0, \dots, n$ can be presented as

$$W = C_{0j} + C_{1j} r^{1+k} + C_{2j} r^{1-k} + \frac{Pr^4}{8(9 - k^2)E_1 D_j} + \frac{B_0 r^2}{2(1 - k^2)E_1 D_j}. \quad (5.26)$$

From (5.26) one easily obtains

$$\begin{aligned}
\frac{dW}{dr} &= C_{1j}(1+k)r^k + C_{2j}(1-k)r^{-k} \\
&+ \frac{Pr^3}{2(9-k^2)E_1D_j} + \frac{B_0r}{(1-k^2)E_1D_j}, \\
\frac{d^2W}{dr^2} &= C_{1j}k(1+k)r^{k-1} + C_{2j}(1-k)(-k)r^{-k-1} \\
&+ \frac{3Pr^2}{2(9-k^2)E_1D_j} + \frac{B_0}{(1-k^2)E_1D_j}.
\end{aligned} \tag{5.27}$$

Substitution of (5.27) in equations (5.15) results in

$$\begin{aligned}
M_1 &= -E_1D_j \left[C_{1j}(1+k)(k+\nu_{21})r^{k-1} + C_{2j}(1-k)(\nu_{21}-k)r^{-k-1} \right. \\
&\quad \left. + \frac{Pr^2(3+\nu_{21})}{2E_1D_j(9-k^2)} + \frac{B_0(1+\nu_{21})}{E_1D_j(1-k^2)} \right], \\
M_2 &= -E_2D_j \left[C_{1j}(1+k)(k\nu_{12}+1)r^{k-1} \right. \\
&\quad \left. + C_{2j}(1-k)(1-k\nu_{12})r^{-k-1} + \frac{Pr^2(1+3\nu_{12})}{2E_1D_j(9-k^2)} + \frac{B_0(1+\nu_{12})}{E_1D_j(1-k^2)} \right]
\end{aligned} \tag{5.28}$$

which hold good for $r \in (a_j, a_{j+1})$, $j = 0, \dots, n$.

The governing equations (1.1) are accompanied with boundary conditions

$$\begin{aligned}
M_1(R) &= 0, \\
W(R) &= 0
\end{aligned} \tag{5.29}$$

and

$$\frac{dW(0)}{dr} = 0 \tag{5.30}$$

in the case of a simply supported plate. If membrane forces are taken into account extra requirements for in-plane quantities are needed.

5.5 Bending of a one-stepped plate

Let us focus our attention on the plate of piecewise constant thickness

$$h = \begin{cases} h_0; & r \in (0, a) \\ h_1; & r \in (a, R). \end{cases} \tag{5.31}$$

For determination of the stress strain state of the plate one can use the relations (5.26) – (5.30) for $n = 1$ and corresponding continuity requirements for the deflection W and the bending moment M_1 .

From physical considerations it is evident that the quantities W , $\frac{dW}{dr}$, M_1 and M_2 must be finite at the center of the plate (for $r = 0$). This is possible only in the case where (here $k > 0$)

$$\begin{aligned} C_{10} &= 0, \\ C_{20} &= 0. \end{aligned} \quad (5.32)$$

Bearing (5.32) in mind one can simplify the notation equalizing

$$\begin{aligned} C_{00} &= W_0, \\ C_{01} &= C, \\ C_{11} &= C_1, \\ C_{21} &= C_2. \end{aligned} \quad (5.33)$$

The continuity of the radial bending moment at $r = a$ leads to the relation

$$C_1(1+k)(k+\nu_{21})a^{k-1} + C_2(1-k)(\nu_{21}-k)a^{-k-1} = 0. \quad (5.34)$$

It is easy to deduce from (5.34)

$$C_2 = -\frac{C_1(1+k)(k+\nu_{21})a^{2k}}{(1-k)(\nu_{21}-k)}. \quad (5.35)$$

Employing the continuity condition for $\frac{dW}{dr}$ to (5.27), (5.30) one has

$$\begin{aligned} C_1(1+k)a^k + C_2(1-k)a^{-k} + \frac{Pa^3}{2(9-k^2)E_1D_1} + \frac{B_0a}{(1-k^2)E_1D_1} \\ = \frac{Pa^3}{2(9-k^2)E_1D_0} + \frac{B_0a}{(1-k^2)E_1D_0}. \end{aligned} \quad (5.36)$$

From (5.36) one can easily determine

$$B_0 = (1-k^2)E_1D_1 \left[\frac{2k(1+k)D_0a^{k-1}C_1}{(k-\nu_{21})(D_1-D_0)} - \frac{Pa^2}{2E_1D_1(9-k^2)} \right] \quad (5.37)$$

The first requirement in (5.29) with (5.28), (5.35) and (5.37) admits to define

$$\begin{aligned} C_1 &= \frac{(k-\nu_{21})(\gamma-1)PR^2[\alpha^2(1+\nu_{21})-3-\nu_{21}]}{2(9-k^2)(1+k)E_1D_1R^{k-1}N}, \\ C_2 &= \frac{(k+\nu_{21})PR^2\alpha^{2k}(\gamma-1)[\alpha^2(1+\nu_{21})-3-\nu_{21}]}{2(9-k^2)E_1D_1R^{-k-1}N} \end{aligned} \quad (5.38)$$

where for the conciseness sake the notation

$$N = (k^2 - \nu_{21}^2)(1 - \alpha^{2k})(\gamma - 1) + 2k(1 + \nu_{21})\alpha^{k-1} \quad (5.39)$$

and

$$\begin{aligned} \gamma &= \frac{D_1}{D_0}, \\ \alpha &= \frac{a}{R} \end{aligned} \quad (5.40)$$

is introduced.

According to the second boundary condition in (5.29) and (5.26) one has

$$C + C_1 R^{k+1} + C_2 R^{1-k} + \frac{PR^4}{8(9 - k^2)E_1 D_1} + \frac{B_0 R^2}{2(1 - k^2)E_1 D_1} = 0. \quad (5.41)$$

The last equation with (5.35), (5.37), (5.40) admits to define

$$\begin{aligned} C &= R^{k+1} C_1 \left[\frac{(1 + k)(k + \nu_{21})\alpha^{2k}}{(1 - k)(\nu_{21} - k)} - \frac{k(1 + k)}{(k - \nu_{21})(\gamma - 1)} - 1 \right] \\ &+ \frac{PR^4(2\alpha^2 - 1)}{8(9 - k^2)E_1 D_1}. \end{aligned} \quad (5.42)$$

Employing the continuity of the deflection W at $r = a$ one obtains

$$\begin{aligned} W_0 &+ \frac{Pa^4}{8(9 - k^2)E_1 D_0} + \frac{B_0 a^2}{2(1 - k^2)E_1 D_0} \\ &= \frac{Pa^4}{8(9 - k^2)E_1 D_1} + \frac{B_0 a^2}{2(1 - k^2)E_1 D_1} + C_1 a^{1+k} + C_2 a^{1-k}. \end{aligned} \quad (5.43)$$

Thus the maximal deflection can be calculated as

$$W_0 = \frac{1 - \gamma}{E_1 D_1} \left[\frac{Pa^4}{8(9 - k^2)} + \frac{B_0 a^2}{2(1 - k^2)} \right] + C_1 a^{k+1} + C_2 a^{1-k} \quad (5.44)$$

where the constants B_0 , C_1 , C_2 , γ and α are defined by (5.37), (5.38), (5.40).

5.6 Optimal design of a stepped plate

The problem posed above consists in the minimization of the maximal deflection $W_0 = W(0)$ under the condition that the material volume of the plate is constrained.

Evidently, the deflection $W(0)$ is a function of design parameters; $W_0 = W_0(a_1, \dots, a_n, h_0, \dots, h_n)$. The constrained minimum of W_0 can be defined employing the extended function

$$I_* = W_0(a_1, \dots, a_n, h_0, \dots, h_n) + \lambda(V - V_0) \quad (5.45)$$

where λ is an unknown Lagrangian multiplier.

The necessary optimality conditions of (5.45) can be written as

$$\begin{aligned}\frac{\partial I_*}{\partial a_i} &= 0; & i &= 1, \dots, n \\ \frac{\partial I_*}{\partial h_j} &= 0; & j &= 0, \dots, n \\ \frac{\partial I_*}{\partial \lambda} &= 0.\end{aligned}\tag{5.46}$$

The system (5.46) consists of $2n + 2$ algebraic equations. These admit to determine $2n + 2$ unknown parameters a_i ($i = 1, \dots, n$), h_j ($j = 0, \dots, n$) and λ for each given volume of the plate V_0 .

5.7 Numerical results and discussion

The non-linear system of equations (5.46) is solved numerically. Calculations are carried out with the aid of computer codes *Python* and *Mathematica*. Calculations are carried out for the plate with radius $R = 1$ m and $h_0 = 0.0025$ m.

Table 5.1: Optimal values of parameters.

V_0	W_o	α	γ	W_*	e
0.45	0.0056	0.5701	0.1852	0.0262	0.2122
0.50	0.0044	0.5475	0.2860	0.0191	0.2306
0.55	0.0039	0.5466	0.3583	0.0143	0.2685
0.60	0.0035	0.5539	0.4230	0.0111	0.3146
0.65	0.0032	0.5670	0.4842	0.0087	0.3678
0.70	0.0030	0.5852	0.5437	0.0070	0.4278
0.75	0.0028	0.6088	0.6028	0.0057	0.4952
0.80	0.0027	0.6384	0.6624	0.0047	0.5709
0.85	0.0026	0.6755	0.7241	0.0039	0.6561
0.90	0.0025	0.7232	0.7904	0.0033	0.7530
0.95	0.0024	0.7887	0.8677	0.0028	0.8651
0.99	0.0024	0.8659	0.9600	0.0025	0.9705

The results of calculations are presented in the Table 5.1 and Fig. 5.1 for the case $n = 1$. The results are obtained for the plate made of a Boron-Aluminium composite consisting of Aluminium matrix and Boron fibers. For this material $E_1 = 204$ GPa, $E_2 = 118$ GPa and $\nu_{12} = 0.27$.

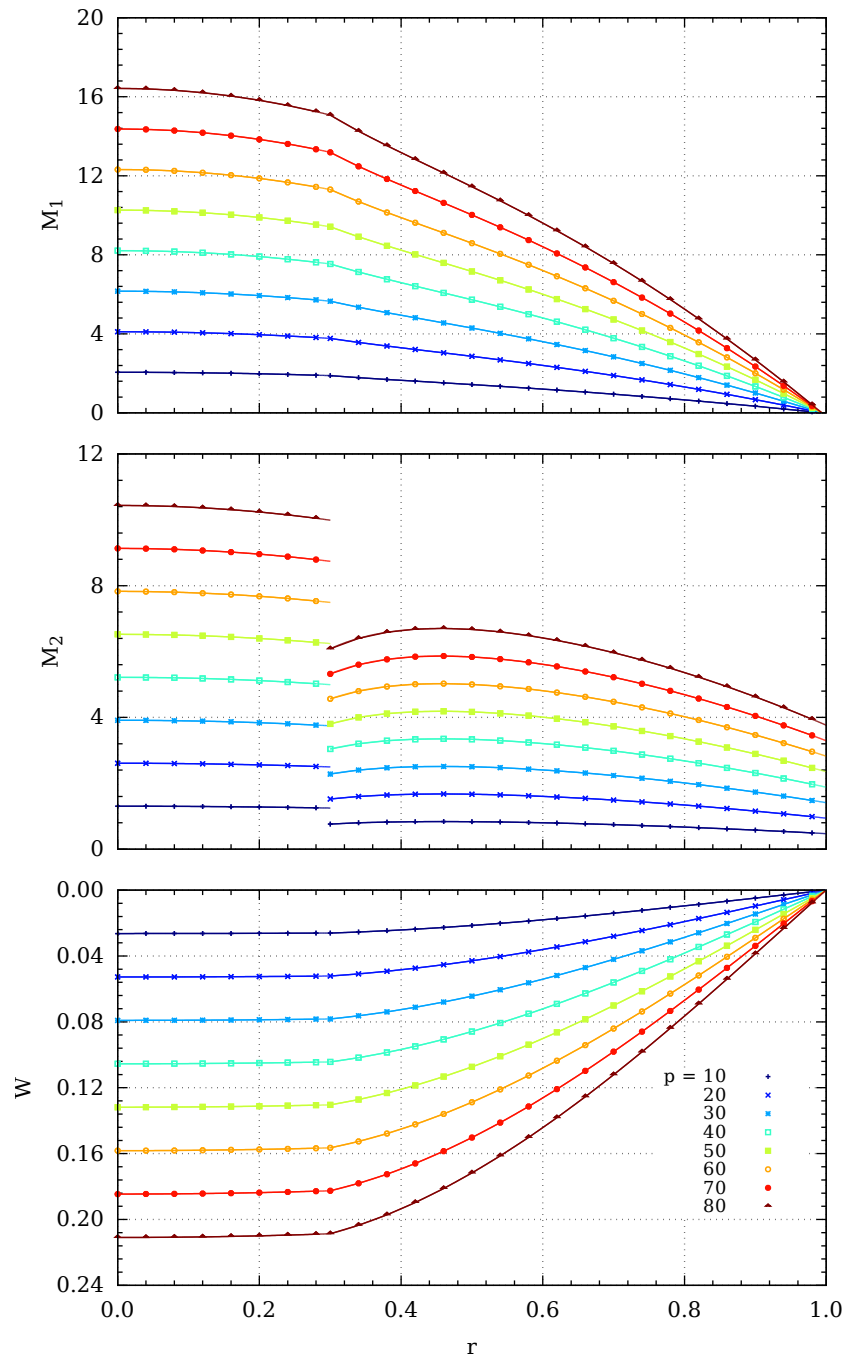


Figure 5.1: Bending moments and deflections.

It can be seen from the middle panel in Fig. 5.1 that the circumferential bending moment is discontinuous at $r = a$, as might be expected. Calculations carried out showed that the jump of M_2 at $r = a$ depends on the ratio of thicknesses in the adjacent sections of the plate.

The radial bending moment M_1 has slope discontinuity at the step position (the top panel in Fig. 5.1) which is less remarkable in the cases of lower load intensities.

The displacements in the transverse direction are presented for different values of the intensity of the transverse pressure on the bottom panel in Fig. 5.1, for different load levels for the stepped plate with the step at $a = 0.3R$.

The efficiency of the design established above is assessed by the ratio

$$e = \frac{W_0}{W_*}, \quad (5.47)$$

W_* being the deflection of the plate of constant thickness $h_* = V_0 h_0$. The values of the coefficient e are accommodated in Table 5.1 together with optimal values α and γ . In the first column of Table 5.1 the values of the plate volume $V_0 = \frac{V}{\pi}$ are presented. It reveals from Table 5.1 that for smaller values of V_0 the deflection of the reference plate of constant thickness W_* is larger, as might be expected. If V_0 increases then the optimal ratio of thicknesses $\gamma = \frac{h_1}{h_0}$ also increases and the step location $a = \alpha R$ moves towards the edge of the plate. However, the maximal deflection of the optimized plate decreases if the quantity V_0 increases.

Calculations carried out showed that the optimal values of α and γ are insensitive with respect to the changes of the load intensity. However, the optimal value of W_0 as well as the quantities W_* and e do depend on the value of the load intensity P .

5.8 Concluding remarks

A method for analysis and optimization of fiber reinforced composite plate was developed. Considering the cases of unidirectional orientations of fibers the composite was modelled as a quasi homogeneous anisotropic material having different properties in different directions. The attention was focused on the cases of radial and circumferential orientation of fibers in the matrix material.

Invoking the methods of non-linear mathematical programming the optimal design parameters of plates with piecewise constant thickness are determined. The optimization problems are solved with the aid unknown Lagrangian multipliers. Numerical results are obtained with the aid of existing computer codes. It was shown that the optimal values of design parameters depend on the physical and geometrical parameters but these are insensitive with respect to the loading level.

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SUMMARY

KOKKUVÕTE

Elastsete-plastsete telgsümmeetriliste plaatide analüüs ja optimeerimine

Käesolevas töös uuritakse ümar- ja rõngasplaate, millele mõjub telgsümmeetriline ristkoormus. Ümarplaat on ringsilinder, mille kõrgus on teiste mõõtmetega võrreldes väike.

Insenerimehaanikast on teada, et konstruktsioonelementide (talade, plaatide, koorikute) modelleerimisel on materjali kokkuhoiuks mõistlik arvestada lisaks elastsetele deformatsioonidele ka plastseid. Esimesed teadusartiklid elastsete-plastsete ümarplaatide kohta ilmusid 1950ndatel B. Tekinalpi [93] ja P. G. Hodge'i [27] sulest.

Käesolevas töös eeldatakse, et vaadeldavad ümar- ja rõngasplaadid on nn *sandwich*-tüüpi ristlõikega materjalist. *Sandwich*-tüüpi ristlõikeks nimetatakse ideaalset kahekihilist ristlõiget, mille kandva kihi paksus h on kihtidevahelise kaugusega H võrreldes väike. Kuna *sandwich*-tüüpi plaadi mudeli korral võib pinged piki paksust lugeda konstantseteks, siis lihtsustab see oluliselt arvutusi. Võime eeldada, et kandva kihi teatud piirkond saab olla kas ainult elastses või täielikult plastses seisundis.

Plastne seisund ja voolavus..

Kui plaadile rakendatakse väikseid koormusi, siis on kogu plaat elastne. Koormuse suurendamisel tekib plaadi sümmetriatelje ümber plastne piirkond. Plastses piirkonnas peab pingeseisundit iseloomustav punkt asuma voolavuspinna (voolavuskõvera) peal. Käesolevas töös kasutatakse tükat linearseid voolavuskõveraid. Plastusteooria põhiseoste kohaselt rahuldab pingeseisund peale voolavuspinna võrrandite ka nn assotseeritud voolavusseadust. Assotseeritud voolavusseaduse kohaselt on deformatsioonikiiruse vektor voolavuspinna igas punktis sellega risti ning suunatud pinnast väljapoole.

Lisaks elastsete-plastsete ümar- ja rõngasplaatide paindeülesannete uurimisele lahendatakse antud töös ka lisatugedega vabalt toetatud ümarplaadi optimeerimise probleem.

Käesolev doktoritöö koosneb viiest peatükist. Esimene peatükk on referatiivne. Selles esitatakse ümarplaadi jaoks tasakaaluvõrrandid, deformatsiooni kiiruse komponendid, plaadi läbipainde diferentsiaalvõrrandid ning selle üldlahend. Samuti tutvustatakse Tresca' voolavustingimust ning assotseeritud voolavusseadust.

Teises peatükis uuritakse konstantse paksusega elastset-plastset servast vabalt toetatud ümarplaati Tresca' ja rombi voolavustingimuse korral. Selgub, et mõlemal juhul saab plastse ja elastse piirkonna rajapunkti tuletada analüütiliselt, st välja arvutada biruutvõrrandist. Selles peatükis lahendatakse numbriliselt ka ühe astmega elastse ümarplaadi optimeerimisülesanne, kus etteantud plaadi ruumala korral arvutatakse

optimaalsed kandvate kihtide paksused ning astme asukoht nii, et plaadi keskpunktis toimuv läbipaine oleks minimaalne. Samuti lahendatakse elastse-plastse konstantse paksusega servast vabalt toetatud rõngasplaadi paindeülesanne, st leitakse erinevate koormuste korral analüütiliselt ja numbriliselt plaadi läbipained ning radiaal- ja tangentsiaalsuunalised paindemomendid.

Kolmandas peatükis uuritakse tükati konstantse paksusega elastset-plastset seest jäigalt kinnitatud rõngasplaati. Selgub, et antud astmelise rõngasplaadi pingeseisundi saab jagada kolme erinevasse staadiumisse. Leitakse läbipaine ja paindemomentide avaldised vastavalt elastse, elastse-plastse ja täiesti plastse plaadi pingeseisundi korral.

Neljandas peatükis lahendatakse analüütiliselt ja numbriliselt lisatugedega servast vabalt toetatud elastse ümarplaadi optimeerimisülesanne. Leitakse ringtugede optimaalsed asukohad nii, et plaadi läbipaine oleks minimaalne.

Viiendas peatükis lahendatakse numbriliselt sarnaselt teise peatüki ülesandele ühe astmega servast vabalt toetatud anisotroopsest materjalist ümarplaadi optimimeerimisülesanne. Selle ülesande lahendamiseks tuletatakse analüütiliselt plaadi tasakaaluvõrrandid, läbipaine ja paindemomentide avaldised anisotroopse materjali korral.

Antud doktoritöö tulemused on publitseeritud seitsmes artiklis [49, 50, 51, 52, 53, 54, 55].

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CURRICULUM VITAE

Personal data

Name	Julia Polikarpus
Date and place of birth	5 May 1981, Tartu, Estonia
Citizenship	Estonian
Address	University of Tartu, Institute of Mathematics J. Liivi 2, Tartu, Estonia
Phone	(+372) 737 6428
E-mail	julia.polikarpus@ut.ee

Education

2005 – 2013	University of Tartu, doctoral studies in mathematics
2005	University of Tartu, Teacher of Mathematics in Upper Secondary School
2003 – 2005	University of Tartu, master's studies in mathematics
1999 – 2003	University of Tartu, bachelor's studies in mathematics
1993 – 1999	Hugo Treffner Gymnasium

Employment

2013 – ...	University of Tartu, Office of Academic Affairs, senior specialist for curriculum development
2011 – ...	University of Tartu, Institute of Mathematics, researcher of theoretical mechanics
2007 – 2011	Tartu Kivilinna Gymnasium, teacher of mathematics
2008 – 2011	University of Tartu, Institute of Mathematics, assistant of theoretical mechanics
2004 – 2008	University of Tartu, Institute of Mathematics, extraordinary assistant of theoretical mechanics
2004 – 2005	Hugo Treffner Gymnasium, teacher of mathematics
2003 – 2005	Tartu Forseliuse Gymnasium, teacher of mathematics

Professional training

02.06 – 11.06.2007	Summer school "Continuum Physics and Engineering Applications" Ráckeve, Hungary
01.10 – 06.10.2007	Autumn school "Structures of the mechanics of complex bodies" Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy

- | | |
|--------------------|--|
| 06.11 – 13.11.2011 | Autumn school "Mathematical Principles for and Advances in Continuum Mechanics" Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy |
| 25.11 – 29.11.2011 | Autumn school "Multi-scale and Multi-field Representations of Condensed Matter Behavior" Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy |

Language skills

Estonian (mother tongue), English, Russian, French

Scientific work

The main fields of interest are elastic, elastic plastic circular and annular plates, optimization of plates. Results have been presented at the following 9 scientific conferences:

"Finnish Mathematical Days 2006 and 2. Finnish-Estonian Mathematical Colloquium" in Tampere, Finland (2006),

"Continuum Physics and Engineering Applications" in Ráckeve, Hungary (2007),

"The 20th EURO Mini Conference Continuous optimization and Knowledge-Based Technologies" in Neringa, Lithuania (2008),

"FinEst Math 2009, 3rd Finnish-Estonian Mathematical Colloquium" in Tartu, Estonia (2009),

"American Conference on Applied Mathematics *Applications of Mathematics & Computer Engineering*" in Puerto Morelos, Mexico (2011),

"International Conference on Optimization and Analysis of Structures" in Tartu, Estonia (2011),

"The 3rd International Conference on Engineering Optimization", in Rio de Janeiro, Brazil (2012),

"The 2nd International Conference on Optimization and Analysis of Structures" in Tartu, Estonia (2013),

"The 3rd International Conference on Optimization and Analysis of Structures" in Tartu, Estonia (2015).

ELULOOKIRJELDUS

Isikuandmed

Nimi	Julia Polikarpus
Sünniaeg ja -koht	5. mai 1981, Tartu, Eesti
Kodakondsus	Eesti
Aadress	Tartu Ülikool, matemaatika instituut J. Liivi 2, Tartu, Eesti
Telefon	(+372) 737 6428
E-post	julia.polikarpus@ut.ee

Haridustee

2005 – 2013	Tartu Ülikool, matemaatika doktoriõpe
2005	Tartu Ülikool, gümnaasiumi matemaatikaõpetaja
2003 – 2005	Tartu Ülikool, matemaatika magistriõpe
1999 – 2003	Tartu Ülikool, matemaatika bakalaureuseõpe
1993 – 1999	Hugo Treffneri Gümnaasium

Teenistuskäik

2013 – ...	Tartu Ülikool, õppeosakond, õppekvaliteedi peaspetsialist
2011 – ...	Tartu Ülikool, matemaatika instituut, teoreetilise mehaanika teadur
2007 – 2011	Tartu Kivilinna Gümnaasium, matemaatikaõpetaja
2008 – 2011	Tartu Ülikool, matemaatika instituut, teoreetilise mehaanika assistent
2004 – 2008	Tartu Ülikool, rakendusmatemaatika instituut, teoreetilise mehaanika erakorraline assistent
2004 – 2005	Hugo Treffneri Gümnaasium, matemaatikaõpetaja
2003 – 2005	Tartu Forseliuse Gümnaasium, matemaatikaõpetaja

Erialane enesetäiendus

02.06 – 11.06.2007	Suvekool "Continuum Physics and Engineering Applications" Ráckeve, Ungari
01.10 – 06.10.2007	Sügiskool "Structures of the mechanics of complex bodies" Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Itaalia
06.11 – 13.11.2011	Sügiskool "Mathematical Principles for and Advances in Continuum Mechanics" Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Itaalia

25.11 – 29.11.2011 Sügiskool "Multi-scale and Multi-field Representations of Condensed Matter Behavior" Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Itaalia

Keelteoskus

eesti keel (emakeel), inglise keel, vene keel, prantsuse keel

Teaduslik tegevus

Peamised uurimisvaldkonnad on elastsed ning elastsed-plastsed ümar- ja rõngasplaadid, nende optimeerimine. Tulemusi on esitatud 9 teaduskonverentsil:

"Finnish Mathematical Days 2006 and 2. Finnish-Estonian Mathematical Colloquium" Tampere Soomes (2006),

"Continuum Physics and Engineering Applications" Ráckeves Ungaris (2007),

"The 20th EURO Mini Conference Continuous optimization and Knowledge-Based Technologies" Neringal Leedus (2008),

"FinEst Math 2009, 3rd Finnish-Estonian Mathematical Colloquium" Tartus Eestis (2009),

Ämerican Conference on Applied Mathematics *Applications of Mathematics & Computer Engineering* Puerto Morelos Mehhikos (2011),

"International Conference on Optimization and Analysis of Structures" Tartus Eestis (2011),

"The 3rd International Conference on Engineering Optimization", Rio de Janeiros Brasiilias (2012),

"The 2nd International Conference on Optimization and Analysis of Structures" Tartus Eestis (2013),

"The 3rd International Conference on Optimization and Analysis of Structures" Tartus Eestis (2015).

LIST OF PUBLICATIONS

1. J. Lellep and J. Polikarpus. Optimization of elastic-plastic circular plates under axisymmetric loading. In L. Sakalauskas, G. W. Weber, and E. K. Zavadskas, editors, *Proceedings of the 20th International Conference "Continuous Optimization and Knowledge-Based Technology"*, pages 291–295, Vilnius, 2008. Technika.
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